Gradient flows: Absolut essentials and manifolds

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Material for a seminar talk

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Abstract: This material is hopefully useful for an understanding of (the first steps of) the "geometry" work of F. Otto from 2001, [1]. We give the easiest nontrivial example for a gradient flow on a manifold.

1. Gradient flow in a Hilbert space

When X is a Hilbert space and $E : X \to \mathbb{R}$ is differentiable, then the differential in a point $u \in X$ is a map $DE(u) : X \to \mathbb{R}$. It is defined as

$$DE(u)\langle v\rangle := \left.\frac{d}{dt}\right|_{t=0} E(u + t\,v) \tag{1.1}$$

for every $v \in X$. With the scalar product on X, we define the gradient $\nabla E(u) \in X$ by demanding

$$DE(u)\langle v\rangle \stackrel{!}{=} \langle \nabla E(u), v\rangle_X \qquad \forall v \in X\,. \tag{1.2}$$

Let us investigate a solution $u : [0, T] \ni t \mapsto u(t) \in X$, of the gradient flow equation

$$\partial_t u = -\nabla E(u)\,, \tag{1.3}$$

which is meant to be satisfied for every $t \in (0, T)$. This equation means that we are "walking in the direction of the steepest decent". We calculate for the evolution of the energy, omitting the argument t,

$$\frac{d}{dt}[E(u)] = DE(u)\langle \partial_t u\rangle = \langle \nabla E(u), \partial_t u\rangle_X < 0\,. \tag{1.4}$$

Moreover, the decay of energy is quantified. The right hand side can be written in any of these forms:

$$-\|\nabla E(u)\|^2 = -\|\partial_t u\|^2 = -\|\partial_t u\|\,\|\nabla E(u)\| = -\frac{1}{2}\|\partial_t u\|^2 - \frac{1}{2}\|\nabla E(u)\|^2\,. \tag{1.5}$$

The subsequent three examples are formal in the sense that (in two of the examples) the energy is not differentiable on the whole space X. Regarding $\Omega \subset \mathbb{R}^n$, we always think of a bounded Lipschitz domain.

1.1. **The L^2-gradient flow of the L^2-energy.** We consider $X = L^2(\Omega)$ and the energy $E : X \to \mathbb{R}$, defined by $E(u) = \frac{1}{2}\int_\Omega |u|^2$. Then

$$DE(u)\langle v\rangle = \int_\Omega u \cdot v\,, \quad \nabla E(u) = u\,. \tag{1.6}$$

The corresponding gradient flow is therefore

$$\partial_t u = -u\,. \tag{1.7}$$

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1.2. **The L^2-gradient flow of the H^1-energy.** We consider $X = L^2(\Omega)$ and the energy $E : X \to \mathbb{R}$, defined by $E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2$. Then

$$DE(u)\langle v\rangle = \int_\Omega \nabla u \cdot \nabla v\,, \quad \nabla E(u) = -\Delta u\,. \tag{1.8}$$

The corresponding gradient flow is therefore

$$\partial_t u = \Delta u\,. \tag{1.9}$$

1.3. **The H^{-1}-gradient flow of the L^2-energy.** We consider $X = H^{-1}(\Omega)$ and the energy $E : X \to \mathbb{R}$, defined by $E(u) = \frac{1}{2}\int_\Omega |u|^2$.

In order to prepare the analysis, we have to choose a scalar product on $H^{-1}(\Omega) = (H^1_0(\Omega))'$. There is a natural choice when we use the scalar product $\langle u, v\rangle_{H^1} := \int_\Omega \nabla u \cdot \nabla v$ on $H^1_0(\Omega)$. Every element $\lambda \in X = H^{-1}(\Omega)$ can be represented by $U_\lambda \in H^1_0(\Omega)$ via $\lambda(\varphi) = \langle U_\lambda, \varphi\rangle_{H^1}$ for all $\varphi \in H^1_0(\Omega)$. By the choice of the scalar product in $H^1_0(\Omega)$, the objects λ and U_λ are related by equation $-\Delta U_\lambda = \lambda$ in the sense of distributions. The natural scalar product on $X = H^{-1}(\Omega)$ is therefore, for $\lambda, \mu \in X$, given by

$$\langle \lambda, \mu\rangle_X := \langle U_\lambda, U_\mu\rangle_{H^1} = \lambda(U_\mu) = \int_\Omega \nabla U_\lambda \cdot \nabla U_\mu = \int_\Omega U_\lambda\, \mu\,, \tag{1.10}$$

where the last expression can only be written in this form when μ is an $L^2(\Omega)$-function.

The differential of E is (formally, when u and v are $L^2(\Omega)$):

$$DE(u)\langle v\rangle = \int_\Omega u \cdot v\,. \tag{1.11}$$

We are now in the position to calculate the gradient $g = \nabla E(u)$. For arbitrary $v \in X$ (in the calculation we actually assume $v \in L^2(\Omega)$):

$$\int_\Omega u \cdot v = DE(u)\langle v\rangle \overset{!}{=} \langle \nabla E(u), v\rangle_X = \langle g, v\rangle_X = \int_\Omega (-\Delta)^{-1}(g)\ v\,. \tag{1.12}$$

Since v was arbitrary, we find $u = (-\Delta)^{-1}(g)$ or, equivalently, $g = -\Delta u$.

The gradient flow corresponding to X and E is therefore

$$\partial_t u = \Delta u\,. \tag{1.13}$$

Even though we have chosen another energy and another underlying space, we have the same equation as in (1.9).

2. Elementary differential geometry

The aim of this section is to support readers that want to understand [1]. Because of this aim, all the notation is as in [1]. There is a minimal exception: We write $DE|_\rho\langle s\rangle$ instead of $\mathrm{diff}E|_\rho.s$.

2.1. **The simplest nontrivial gradient flow on a manifold.** Guiding question: Given a manifold $\mathcal{M}$ (the elements are denoted by ρ) and given a function $E : \mathcal{M} \to \mathbb{R}$, we are interested in the gradient flow equation

$$\partial_t \rho = -\nabla E(\rho)\,. \tag{2.1}$$

More precisely: We seek a map $\rho : [0, T] \to \mathcal{M}$ such that (2.1) holds for almost every t.

We must ask: What exactly is meant with (2.1)? In particular: What kind of object is the gradient? We recall here some differential geometry in order to clearify these questions. With the example of $\mathcal{M} = S^1 \subset \mathbb{R}^2$, we illustrate the concepts.

Abstract manifold setting	Our example
Hilbert space H	$H = \mathbb{R}^2$
manifold $\mathcal{M} \subset H$	Our choice: $\mathcal{M} := S^1$
elements $\rho \in \mathcal{M}$	$S^1 = \{\rho \in \mathbb{R}^2 \mid \rho_1^2 + \rho_2^2 = 1\}$
tangent space $T_\rho\mathcal{M}$	$T_\rho S^1 = \{s \in \mathbb{R}^2 \mid s \cdot \rho = 0\}$
equivalent curves γ with $\gamma(0) = \rho$	subset of vectors $\gamma'(0) \in H$
metric tensor $g = g_\rho(.,.)$	$g_\rho(s_1, s_2) = \langle s_1, s_2\rangle_H = s_1 \cdot s_2$
Riemannian manifold: part of the definition of $\mathcal{M}$	submanifold of a Hilbert space: metric induced by H
energy functional $E : \mathcal{M} \to \mathbb{R}$	Our choice: $E(\rho) := \rho_2$ (height)
differential $DE\|_\rho : T_\rho\mathcal{M} \to \mathbb{R}$	$DE\|_\rho\langle s\rangle = \frac{d}{dt}(E \circ \gamma)(t_0)$
	$\gamma(t_0) = \rho$, $\gamma'(t_0) = s$

Table 1. Embedded manifold and our example

We consider the example that is outlined in the right part of Table 1. Let us calculate the differential of E in the point $\rho = (\cos(t_0), \sin(t_0))$. An arbitrary tangential vector in ρ is of the form $\mu\,(-\sin(t_0), \cos(t_0))$. It is sufficient to evaluate $DE|_\rho\langle s\rangle$ for $s = (-\sin(t_0), \cos(t_0))$. As a curve through ρ with derivative s we choose $\gamma(t) := (\cos(t), \sin(t))$. We find

$$DE|_\rho\langle s\rangle = \frac{d}{dt}(E \circ \gamma)(t)|_{t=t_0} = \frac{d}{dt}\sin(t)|_{t=t_0} = \cos(t_0)\,.$$

The action on an arbitrary tangential vector is therefore, for $\rho = (\cos(t_0), \sin(t_0))$,

$$DE|_\rho\langle \mu\,(-\sin(t_0), \cos(t_0))\rangle = \mu\,\cos(t_0)\,. \tag{2.2}$$

With this calculation, DE is determined.

Definition 2.1 (Gradient)**.** *The vector $\nabla E(\rho)$ is the element of $T_\rho\mathcal{M}$ such that*

$$g_\rho(\nabla E(\rho), s) = DE|_\rho\langle s\rangle \tag{2.3}$$

holds for every $s \in T_\rho\mathcal{M}$.

We continue our example, we now consider a point $\rho = (\cos(t), \sin(t))$. The gradient $\nabla E(\rho)$ is a tangent vector, therefore, for some $\lambda \in \mathbb{R}$, there must hold $\nabla E(\rho) = \lambda\,(-\sin(t), \cos(t))$. Test vectors are also tangential vectors, we write them as $s = \mu\,(-\sin(t), \cos(t)) \in T_\rho\mathcal{M}$. The gradient is defined by

$$\lambda\,\mu = g_\rho(\nabla E(\rho), s) \overset{!}{=} DE|_\rho\langle s\rangle = \mu\,\cos(t)\,. \tag{2.4}$$

Since this should hold for every $\mu \in \mathbb{R}$, we find $\lambda = \cos(t)$. We have thus determined the gradient $\nabla E(\rho)$:

$$\nabla E((\cos(t), \sin(t))) = \cos(t)\,\,(-\sin(t), \cos(t))\,. \tag{2.5}$$

This result coincides with intuition — at least when the intuition is well trained: The gradient is obtained from the gradient of E in the ambient space (which is e_2), by a projection onto the tangential space.

Without loss of generality, we can write the solution $\rho = \rho(t)$ of (2.1) as $\rho(t) = (\cos(\psi(t)), \sin(\psi(t))$. This is true since every point on the manifold can be written as $(\cos(\psi), \sin(\psi)$ for some $\psi \in \mathbb{R}$. We note in passing: ψ is called the lifting of ρ (Deutsch: "Liftung").

With this notation, the gradient flow equation (2.1) reads

$$\psi'(t)\,(-\sin(\psi(t),\cos(\psi(t))) = -\cos(\psi(t))\ (-\sin(\psi(t)),\cos(\psi(t)))\,. \tag{2.6}$$

The equation for ψ is therefore

$$\psi'(t) = -\cos(\psi(t))\,. \tag{2.7}$$

2.2. **A manifold that is given by a submersion.** In [1], the relevant manifold for the gradient flow is not given as a submanifold of a Hilbert-space or Banach-space. Instead, the Riemannian manifold is "parametrized" with a submersion. We want to illustrate also this concept with a simple example. We use a flat manifold $\mathcal{M}^*$ and regard $\mathcal{M}$ as the image of $\mathcal{M}^*$ under some submersion Φ. We note that, in this outline, it is not relevant that $\mathcal{M}^*$ is flat. Our interest here is not the gradient flow, but the "right" Riemannian metric of $\mathcal{M}$.

A *submersion* is a differentiable map whose differential is everywhere surjective (while, for an *immersion,* the differential is everywhere injective).

Manifold defined by a submersion	Our example
flat manifold $\mathcal{M}^*$	$\mathcal{M}^* = \mathbb{R}^3$
elements $\Phi \in \mathcal{M}^*$	$\Phi = (\Phi_1, \Phi_2, \Phi_3)$
submersion $\Pi : \mathcal{M}^* \to \mathcal{M}$	$\Pi : \Phi \mapsto (\cos(\Phi_1), \sin(\Phi_1))$
tangent space $T_\Phi\mathcal{M}^*$	$T_\Phi\mathbb{R}^3 = \mathbb{R}^3$
metric tensor $g^*_\Phi(v_1, v_2)$	$g^*_\Phi(v_1, v_2) = \langle v_1, v_2\rangle_{\mathbb{R}^3}$
tangential $T_\Phi\Pi : T_\Phi\mathcal{M}^* \to T_\rho\mathcal{M}$	$T_\Phi\Pi\langle(\lambda_1,\lambda_2,\lambda_3)\rangle = \lambda_1\,(-\sin(\Phi_1), \cos(\Phi_1))$
where $\rho = \Pi(\Phi)$	where $\rho = (\cos(\Phi_1), \sin(\Phi_1)$
kernel $\ker T_\Phi\Pi$	$\ker T_\Phi\Pi = \{0\} \times \mathbb{R}^2$

TABLE 2. Abstract submersion and the example. Our example continues, we still consider $\mathcal{M} = S^1 \subset \mathbb{R}^2$.

Definition 2.2 (Isometric submersion)**.** *The submersion is an isometric submersion if the following relation holds between g^* and g (base points are connected via $\rho = \Pi(\Phi)$):*

$$g_\rho(s,s) = \inf\left\{g^*_\Phi(v,v)\,|\,T_\Phi\Pi\langle v\rangle = s\right\}\,. \tag{2.8}$$

Let us check this property in our example: Vectors $v \in T_\Phi\mathcal{M}^*$ are vectors $v = (v_1, v_2, v_3) \in \mathbb{R}^3$. Vectors $s \in T_\rho\mathcal{M}$ are of the form $s = \lambda\,(-\sin(\Phi_1), \cos(\Phi_1))$. The condition $T_\Phi\Pi\langle v\rangle = s$ reads $v_1\,(-\sin(\Phi_1), \cos(\Phi_1)) = \lambda\,(-\sin(\Phi_1), \cos(\Phi_1))$, which is equivalent to $v_1 = \lambda$. Relation (2.8) therefore demands for every $s = \lambda\,(-\sin(\Phi_1), \cos(\Phi_1))$:

$$\lambda^2 = g_\rho(s,s) \overset{!}{=} \inf\left\{g^*_\Phi(v,v)\,|\,T_\Phi\Pi\langle v\rangle = s\right\} = \inf\left\{|v|^2_{\mathbb{R}^3}\,\middle|\,v_1 = \lambda\right\} = \lambda^2\,. \tag{2.9}$$

We see that equality holds for all $\lambda \in \mathbb{R}$. Therefore, in our example, the map Π is indeed an isometric submersion.

References

[1] F. Otto. The geometry of dissipative evolution equations: the porous medium equation. *Comm. Partial Differential Equations*, 26(1-2):101–174, 2001.