Dispersion of waves in heterogeneous media

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Physical origins of the wave equation

Acoustics

Pressure in an ideal gas

$$\rho_0 \partial_t v + p_0'(\rho_0)\,\nabla\rho = 0$$
$$\partial_t \rho + \rho_0 \nabla \cdot v = 0$$

Light

Maxwell's equations

$$\partial_t(\mu H) = -\mathrm{curl}\ E$$
$$\partial_t(\varepsilon E) = \mathrm{curl}\ H$$

Elastic media

Equations of elasticity

$$\rho \partial_t^2 u + \nabla \cdot \sigma = 0$$
$$\sigma = A \nabla^s u$$

In simplified settings, each model leads to the

Wave equation

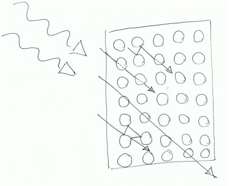
$$\rho \partial_t^2 u = \nabla \cdot (a \nabla u)$$

with coefficients $\rho = \rho(x)$ and $a = a(x)$

Assumptions:

- Polarised H- and E-field, $H = u(x_1, x_2)e_3$
- Constant coefficients
- Uniaxial deformation $u = u(x_1, x_2)e_3$

Heterogeneous media



Let $a : \mathbb{R}^n \to (\delta, \infty)$ be 1-periodic, set $a_\varepsilon(x) := a(x/\varepsilon)$

$$\partial_t^2 u^\varepsilon = \nabla \cdot (a(x/\varepsilon) \nabla u^\varepsilon(x))$$

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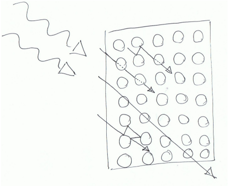

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Classical homogenization): $u^\varepsilon \approx u$, where u solves

$$\partial_t^2 u(x,t) = (c^*)^2 \partial_x^2 u(x,t)\,, \qquad u(x,0) = f(x)\,, \quad \partial_t u(x,0) = 0$$

Exact solution: $u(x,t) = \frac{1}{2} f(x - c^* t) + \frac{1}{2} f(x + c^* t)$

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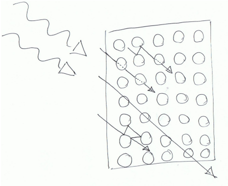

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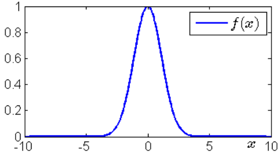
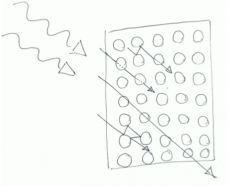


Initial values f

Heterogeneous media



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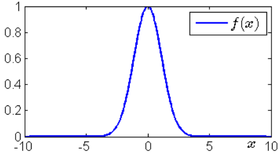

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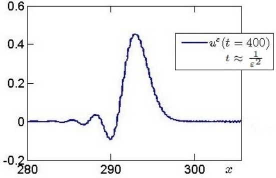

Numerical solution for $\varepsilon = 1/20$

Dispersion

Ansatz: $u(x,t) = U(x)e^{-\mathrm{i}\omega t}$ with $U(x) = e^{\mathrm{i}k\cdot x}$

$$\partial_t^2 u = c^2 \Delta u \iff \omega^2 = c^2|k|^2$$

Dispersion relation: $\omega(k) = \pm c|k|$

Solutions: (here: $x \in \mathbb{R}$)

$$u(x,t) = \int_{\mathbb{R}} \hat{f}_{\pm}(k) e^{\mathrm{i}kx \pm \mathrm{i}c|k|t}\, dk$$

- Choosing $\hat{f}_{\pm}$ appropriately, we can satisfy initial conditions
- Observe: solutions *depend only on* $x + ct$ and on $x - ct$

Dispersion

... describes the effect that harmonic waves travel at different speeds. This occurs iff $k \mapsto \omega(k)$ is not 1-homogeneous.

The homogenization problem

Waves propagate in a **periodic medium**, periodicity length $\varepsilon > 0$.

Variables

displacement field	$u^\varepsilon : \mathbb{R}^n \times (0, T_\varepsilon) \to \mathbb{R}$
elastic modulus	$a_Y : \mathbb{R}^n \to \mathbb{R}^{n\times n}$
initial displacement	$f : \mathbb{R}^n \to \mathbb{R}$

Heterogeneous Wave Equation

$$\partial_t^2 u^\varepsilon(x,t) = \nabla \cdot \left(a_Y \left(\tfrac{x}{\varepsilon} \right) \nabla u^\varepsilon(x,t) \right)$$
$$u^\varepsilon(x,0) = f(x)\,, \quad \partial_t u^\varepsilon(x,0) = 0$$

- $a_Y(\cdot) \in C^\infty(\mathbb{R}^n, \mathbb{R}^{n\times n})$ periodic for the cube $Y := (-\pi,\pi)^n$, i.e. $a_Y(y) = a_Y(y + 2\pi e_i)$
- $a_Y(y)$ symmetric and positive definite matrix: $a_Y(y)_{ij} = a_Y(y)_{ji}$ and $\sum_{i,j=1}^n (a_Y(y))_{ij}\xi_i\xi_j \geq \gamma|\xi|^2$
- f smooth

Question: What is the effective behavior of u^ε as $\varepsilon \to 0$?

Main result: $T_\varepsilon = T\varepsilon^{-2}$

We derive an effective equation for large times!

Weakly dispersive effective equation

$$\partial_t^2 w^\varepsilon = AD^2 w^\varepsilon + \varepsilon^2 ED^2 \partial_t^2 w^\varepsilon - \varepsilon^2 FD^4 w^\varepsilon$$
$$w^\varepsilon(x,0) = f(x)\,, \quad \partial_t w^\varepsilon(x,0) = 0$$

Constant coefficients:

$A, E \in \mathbb{R}^{n\times n}$ and $F \in \mathbb{R}^{n\times n\times n\times n}$

$$\begin{aligned} AD^2 &:= \sum A_{ij}\partial_i\partial_j\,, \\ ED^2 &:= \sum E_{ij}\partial_i\partial_j\,, \\ FD^4 &:= \sum F_{ijml}\partial_i\partial_j\partial_m\partial_n \end{aligned}$$

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Theorem (Dohnal, Lamacz, B.S., 2014 and 2015)

There exist $A, E \in \mathbb{R}^{n\times n}$ and $F \in \mathbb{R}^{n\times n\times n\times n}$, computable from a_Y, s.t.

1. the weakly dispersive effective equation is well-posed
2. there holds the error estimate

$$\sup_{t\in[0,T\varepsilon^{-2}]} \|u^\varepsilon(\cdot,t) - w^\varepsilon(\cdot,t)\|_{L^2(\mathbb{R}^n)+L^\infty(\mathbb{R}^n)} \le C_0\varepsilon.$$

Norm: $\|u\|_{X+Y} := \inf\{\|u_1\|_X + \|u_2\|_Y \,:\, u = u_1 + u_2\}$

Bloch wave analysis: Basics

1.) $f : \mathbb{R}^n \to \mathbb{R}$ is written with a Fourier transform:

$$f(x) = \int_{\mathbb{R}^n} \hat{f}(\xi) e^{\mathrm{i}\xi \cdot x} \, d\xi$$

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$$f(x) = \int_Z \underbrace{\sum_k \hat{f}(k+\Theta) e^{\mathrm{i}k\cdot x}}_{=:F}\, e^{\mathrm{i}\Theta\cdot x}\, d\Theta$$

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3.) $F = F(x;\Theta)$ is expanded in periodic eigenfunctions $\Phi_m(x;\Theta)$:

$$F(x;\Theta) = \sum_{m\in\mathbb{N}} \alpha_m(\Theta)\Phi_m(x;\Theta)$$

$\Psi_m(x;\Theta) = \Phi_m(x;\Theta)e^{\mathrm{i}\Theta\cdot x}$ solves

$$-\nabla\cdot(a(x)\nabla\Psi_m(x)) = \mu_m(\Theta)\,\Psi_m(x)$$

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Result: The operator $L = -\nabla\cdot(a(.)\nabla)$ acts as a multiplier:

$$Lf = L\int_Z \sum_{m\in\mathbb{N}} \alpha_m(\Theta)\Phi_m(x;\Theta)e^{\mathrm{i}\Theta\cdot x}\, d\Theta = \int_Z \sum_{m\in\mathbb{N}} \alpha_m\mu_m\Phi_m(x)e^{\mathrm{i}\Theta\cdot x}\, d\Theta$$

Bloch analysis I

Bloch-transform f with basis functions w_m^ε and coefficients $\hat{f}_m^\varepsilon(k)$

Step 1: The solution u^ε of $\partial_t^2 u^\varepsilon = -L_\varepsilon u^\varepsilon$ can be represented as

$$u^\varepsilon(x,t) = \sum_{m=0}^{\infty} \int_{Z/\varepsilon} \hat{f}_m^\varepsilon(k)\, w_m^\varepsilon(x,k)\, \mathrm{Re}\left(e^{\mathrm{i}t\sqrt{\mu_m^\varepsilon(k)}} \right)\, dk$$

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$$\sup_{t\in(0,\infty)} \left\| \sum_{m=1}^{\infty} \int_{Z/\varepsilon} \hat{f}_m^\varepsilon(k)\, w_m^\varepsilon(x,k)\, \mathrm{Re}\left(e^{\mathrm{i}t\sqrt{\mu_m^\varepsilon(k)}}\right)\, dk \right\|_{L^2(\mathbb{R}^n)} \leq C_0\varepsilon$$

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Step 3: Let f be

$$f(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} F_0(k) e^{\mathrm{i}k\cdot x}\, dk$$

with $F_0 : \mathbb{R}^n \to \mathbb{C}$ supported on $K \subset\subset \mathbb{R}^n$. Then

$$\|\hat{f}_0^\varepsilon - F_0\|_{L^1(Z/\varepsilon)} \le C_0\varepsilon$$

Bloch analysis II

Step 4: Taylor expansion of the rescaled eigenvalue

$$\mu_0^\varepsilon(k) = \frac{1}{\varepsilon^2}\mu_0(\varepsilon k) = \sum\sum A_{lm}k_l k_m + \varepsilon^2 \sum C_{lmnq}k_l k_m k_n k_q + O(\varepsilon^4)$$

For the square root we use $\sqrt{a+c} = \sqrt{a} + \frac{1}{2\sqrt{a}}c + O(|c|^2)$

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Proposition (Bloch-wave approximation of u^ε)

$$v^\varepsilon(x,t) := (2\pi)^{-n/2}\,\frac{1}{2}\sum_{\pm}\int_K F_0(k)e^{\mathrm{i}k\cdot x}\exp\left(\pm\mathrm{i}t\sqrt{\sum A_{lm}k_l k_m}\right)$$

$$\times\exp\left(\pm\frac{\mathrm{i}\varepsilon^2}{2}t\frac{\sum C_{lmnq}k_l k_m k_n k_q}{\sqrt{\sum A_{lm}k_l k_m}}\right)dk$$

$$\sup_{t\in[0,T\varepsilon^{-2}]}\|u^\varepsilon(\cdot,t) - v^\varepsilon(\cdot,t)\|_{(L^2+L^\infty)(\mathbb{R}^n)} \leq C_0\varepsilon$$

Recall: $\|u\|_{X+Y} := \inf\{\|u_1\|_X + \|u_2\|_Y \;:\; u = u_1 + u_2\}$

Decomposition lemma

The (formal) equation for v is the **“bad Boussinesq equation”**

$$\boxed{\partial_t^2 v(x,t) = AD^2 v(x,t) - \varepsilon^2 C D^4 v(x,t)}$$

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Idea: transform in a well-posed equation with a **replacement trick**

1. rewrite last term as $-CD^4 = ED^2 AD^2$
   for a symmetric, positive semidefinite matrix $E \in \mathbb{R}^{n\times n}$
2. replace AD^2 by ∂_t^2 to obtain the well-posed equation
   $$\partial_t^2 w^\varepsilon(x,t) = AD^2 w^\varepsilon(x,t) + \varepsilon^2 ED^2 \partial_t^2 w^\varepsilon(x,t)$$

It is possible that such a matrix E does not exist!

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Lemma (Decomposability)

Let $A \in \mathbb{R}^{n\times n}$ be symmetric and positive definite. Let $C \in \mathbb{R}^{n\times n\times n\times n}$. There exist symmetric and positive semidefinite $E \in \mathbb{R}^{n\times n}$ and $F \in \mathbb{R}^{n\times n\times n\times n}$ such that

$$-CD^4 = ED^2 AD^2 - FD^4$$

Proof of the error estimate

We have:

1. $\sup_{t\in[0,T\varepsilon^{-2}]} \|u^\varepsilon(\cdot,t) - v^\varepsilon(\cdot,t)\|_{(L^2+L^\infty)(\mathbb{R}^n)} \leq C_0\varepsilon$
2. $\partial_t^2 v^\varepsilon = AD^2 v^\varepsilon + \varepsilon^2 ED^2\partial_t^2 v^\varepsilon - \varepsilon^2 FD^4 v^\varepsilon + O(\varepsilon^4)$

Error estimate

Let u^ε be the solution to the heterogeneous wave equation. Let w^ε be a solution to the weakly dispersive effective equation

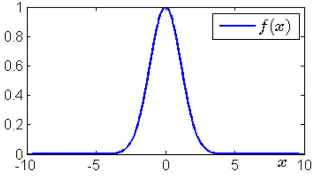
$$\partial_t^2 w^\varepsilon(x,t) = AD^2 w^\varepsilon(x,t) + \varepsilon^2 ED^2\partial_t^2 w^\varepsilon(x,t) - \varepsilon^2 FD^4 w^\varepsilon(x,t).$$

Then

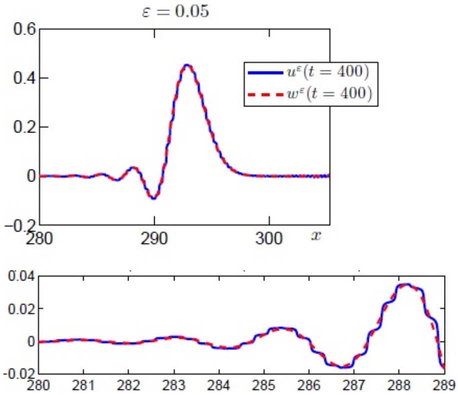
$$\sup_{t\in[0,T\varepsilon^{-2}]} \|u^\varepsilon(\cdot,t) - w^\varepsilon(\cdot,t)\|_{(L^2+L^\infty)(\mathbb{R}^n)} \leq C_0\varepsilon.$$

Proof: Testing procedure to compare (derivatives of) w^ε and v^ε. Triangle inequality and interpolation lemma.

1-dimensional numerical results



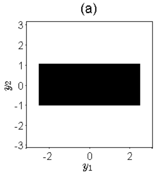
Initial datum f



Comparison with w^ε

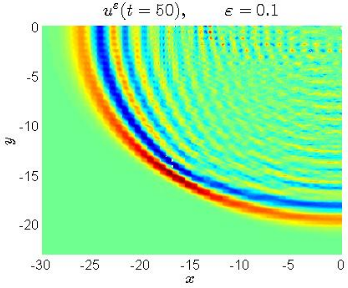
$$\partial_t^2 w^\varepsilon = AD^2 w^\varepsilon + \varepsilon^2 ED^2 \partial_t^2 w^\varepsilon - \varepsilon^2 FD^4 w^\varepsilon$$

2-dimensional numerical results



Numerical comparison:

- the original problem with ε-scale
- the dispersive equation



u^{ε}: solution for coefficient a_{ε}

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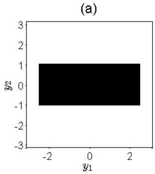

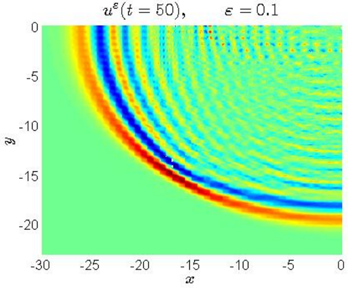

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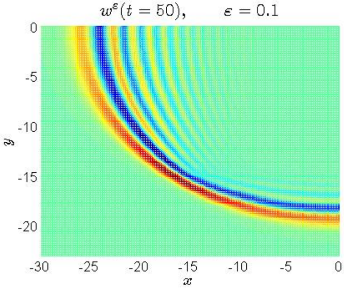


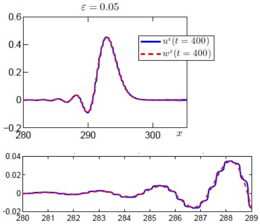
w^ε: solution of weakly dispersive equation

$$\partial_t^2 w^\varepsilon = AD^2 w^\varepsilon + \varepsilon^2 ED^2 \partial_t^2 w^\varepsilon - \varepsilon^2 FD^4 w^\varepsilon$$

Conclusions

- **Finite time:** The original problem is approximated well by the effective wave equation (no dispersion)
- **Long time $t \in (0, T\varepsilon^{-2})$:** Dispersive effects occur! They are effectively described by the weakly dispersive model

Methods: Bloch waves, replacement trick, energy estimates and interpolation



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Thank you!