The time horizon for stochastic homogenization of the one-dimensional wave equation

Wave phenomena conference 2025

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Wave equation in one dimension

Aim

Wave equation with unknown u^ε in one space dimension

$$\rho_\varepsilon \partial_t^2 u^\varepsilon - \partial_x \left(a_\varepsilon \partial_x u^\varepsilon\right) = f_\varepsilon$$

for a given right hand side f_ε, homogeneous initial conditions

Coefficients are random with length scale parameter $\varepsilon > 0$,

$$\rho_\varepsilon(x) := \rho(x/\varepsilon) \quad \text{and} \quad a_\varepsilon(x) := a(x/\varepsilon)$$

Traditional homogenization

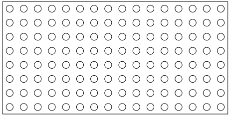
$u^\varepsilon \to \bar{u}$ for $\varepsilon \to 0$ on every time interval, $t \in [0, T_0]$

$\bar{u}$ solves a wave equation with constant coefficients $\bar{\rho}$ and $\bar{a}$:

$$\bar{\rho} \partial_t^2 \bar{u} - \partial_x \left(\bar{a} \partial_x \bar{u}\right) = \bar{f}$$

Periodic case: ρ and a are 1-periodic, $\bar{\rho} = \fint \rho$ and $\bar{a}^{-1} = \fint a^{-1}$

Deterministic: Homogenization problem



Let $a : \mathbb{R}^d \to (\delta, \infty)$ be 1-periodic (in each direction). Set $a_\varepsilon(x) := a(x/\varepsilon)$

Homogenization problem

$$\partial_t^2 u^\varepsilon = \nabla \cdot (a_\varepsilon(x) \nabla u^\varepsilon(x))$$

Classical homogenization: $u^\varepsilon \approx \bar{u}$, where $\bar{u}$ solves

Homogenized equation

$$\partial_t^2 \bar{u}(x,t) = \nabla \cdot a_* \nabla \bar{u}(x,t)$$

Dimension 1:

Wave equation: $\partial_t^2 u(x,t) = c^2 \partial_x^2 u(x,t)$

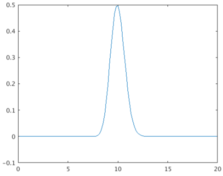
Initial conditions: $u(x,0) = g(x)$, $\partial_t u(x,0) = 0$

An exact solution is given by

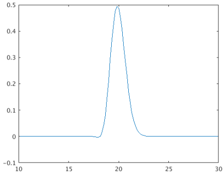
$$u(x,t) = \frac{1}{2} g(x - ct) + \frac{1}{2} g(x + ct)$$

Deterministic: Dispersion in heterogeneous media

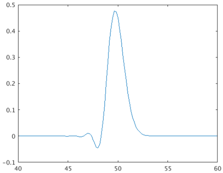
Initial condition: $u_0(x) = g(x) = e^{-|x|^2}$
Solution of homogenized equation $\partial_t^2 u = \Delta u$ is exact shift
Plotted are the solutions to the ε-problem (wave equation), $\varepsilon = 1/6$



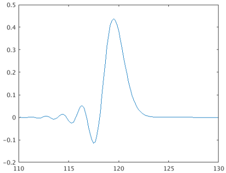
Time $t = 10$. Right-going pulse centered at position $x = 10$



At time $t = 20$, the form is essentially unchanged



At time $t = 50$



At time $t = 120$

J. Fish, W. Chen, and G. Nagai. Nonlocal dispersive model for wave propagation in heterogeneous media, 2002.

Deterministic: Two effective PDEs

A formal calculation suggests to replace the ε-wave equation by

$$\partial_t^2 u = \partial_x^2 u + \frac{\varepsilon^2}{12}\partial_x^4 u$$

Name: "Bad Boussinesq equation" (Santosa-Symes) It is ill-posed!

Replacement trick

Since highest order is $\partial_t^2 u = \partial_x^2 u$, we may write also

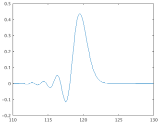
$$\partial_t^2 u = \partial_x^2 u + \frac{\varepsilon^2}{12}\partial_x^2\partial_t^2 u$$

Theorem: This is a good approximation on time intervals $[0, T_0\varepsilon^{-2}]$

Long time behavior of

$$\partial_t^2 u = c^2\partial_x^2 u + \varepsilon^2 p\,\partial_x^4 u$$

We want an equation for *this* $\longrightarrow$



Solution at time $t = 120$

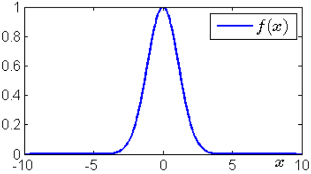
Result: Linearized KdV-equation

$$\partial_\tau V(z,\tau) = -\frac{p}{2c}\,\partial_z^3 V(z,\tau)$$

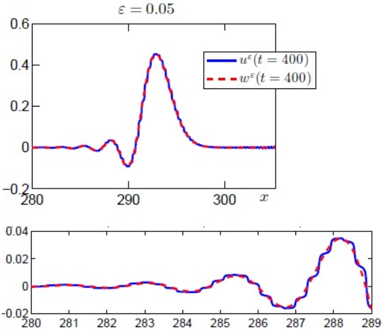
Deterministic: 1-dimensional numerical results

Numerical comparison:

- the original problem with ε-scale
- the weakly dispersive equation



Initial datum f



Comparison with w^ε

A.Lamacz. Dispersive effective models for waves in heterogeneous media, 2011.

T.Dohnal, A.Lamacz, B.S. Bloch-wave homogenization on large time scales and dispersive effective wave equations, 2014.

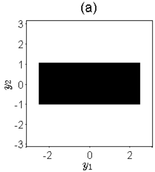
A.Abdulle, T.Pouchon. Effective models for the multidimensional wave equation in heterogeneous media over long time and numerical homogenization, 2016.

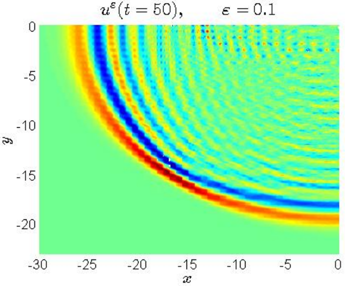
A.Benoit, A.Gloria. Long-time homogenization and asymptotic ballistic transport of classical waves, 2017.

Deterministic: 2-dimensional numerical results

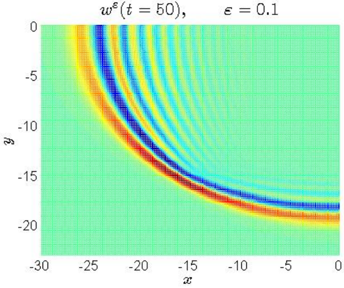
Numerical comparison:

- the original problem with ε-scale
- the weakly dispersive equation





u^ε: solution for coefficient a_ε



w^ε: solution of weakly dispersive equation

$$\partial_t^2 w^\varepsilon = AD^2 w^\varepsilon + \varepsilon^2 ED^2 \partial_t^2 w^\varepsilon - \varepsilon^2 FD^4 w^\varepsilon$$

Stochastic model

For a probability space $(\Omega_{\mathcal{P}}, \mathcal{A}, \mathcal{P})$ and $\Lambda > 0$ we are given maps

$$\rho : \mathbb{R} \times \Omega_{\mathcal{P}} \to [\Lambda^{-1}, \Lambda] \quad \text{and} \quad a : \mathbb{R} \times \Omega_{\mathcal{P}} \to [\Lambda^{-1}, \Lambda]$$

$\langle \cdot \rangle$ denotes the expectation, we do not demand *ergodicity*

Definition (Effective coefficients)

$$\bar{\rho} := \lim_{|r| \to \infty} \left\langle ⨍_0^r ⨍_0^y \rho(s)\, ds\, dy \right\rangle , \quad \bar{a} := \lim_{|r| \to \infty} \left\langle ⨍_0^r ⨍_0^y 1/a(s)\, ds\, dy \right\rangle^{-1}$$

Task

Compare the solutions $u^\varepsilon : \mathbb{R} \times [0, \infty) \times \Omega_{\mathcal{P}} \to \mathbb{R}$ of

$$\rho_\varepsilon \partial_t^2 u^\varepsilon - \partial_x \left(a_\varepsilon \partial_x u^\varepsilon \right) = f$$

and $\bar{u} : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ of

$$\bar{\rho} \partial_t^2 \bar{u} - \partial_x \left(\bar{a} \partial_x \bar{u} \right) = f$$

Both with trivial initial conditions, e.g.: $u^\varepsilon(\cdot, 0) = \partial_t u^\varepsilon(\cdot, 0) = 0$

Correctors

Definition (Correctors)

Let $\rho, a : \mathbb{R} \times \Omega_{\mathcal{P}} \to (0, \infty)$ be stochastic coefficients, $\bar{\rho}, \bar{a}$ real numbers
Correctors:

$$\Psi(y, \omega) := \int_0^y \left\{ \frac{\rho(s)}{\bar{\rho}} - 1 \right\} ds \,, \quad \Phi(y, \omega) := \int_0^y \left\{ \frac{\bar{a}}{a(s)} - 1 \right\} ds$$

(we suppress the argument ω in ρ and a, below also in Ψ and Φ)

Model class parameter (loose definition)

The model class parameter is the value $\gamma \in [0, 1]$ such that,
for some $C > 0$ and all r,

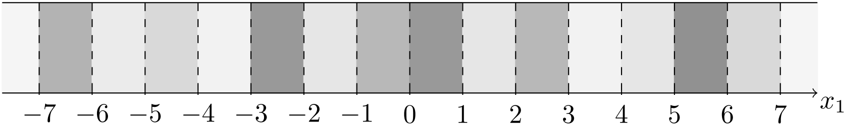
$$\left\langle \fint_0^r |\Psi(y)|^2 \, dy \right\rangle + \left\langle \fint_0^r |\Phi(y)|^2 \, dy \right\rangle \le C(1 + |r|^\gamma)^2$$

- Deterministic medium: No growth, $\gamma = 0$
- Stochastic i.i.d. medium: $\gamma = 1/2$
- Very non-ergodic medium: $\gamma = 1$

γ is the growth-rate of the correctors
$\Phi(r) \sim r^\gamma$

Important example: Stochastic i.i.d. medium

Let $(a_j)_{j\in\mathbb{Z}}$ and $(\rho_j)_{j\in\mathbb{Z}}$ be i.i.d. random variables
e.g.: uniform distribution in $[1, 2]$



Set $a(x) = a_j$ and $\rho(x) = \rho_j$ for $x \in [j, j+1)$

Properties of the i.i.d. model

- The averages are $\bar{\rho} = \langle \rho_0 \rangle$ and $\bar{a} := \langle a_0^{-1} \rangle^{-1}$
- The model class parameter of this model is $\gamma = 1/2$

Sketch of proof: Consider natural numbers y
$\Phi(y)$ is a sum of y i.i.d. random variables with vanishing expected value
Therefore, the variance is

$$\langle |\Phi(y)|^2 \rangle = |y| \sigma^2\,, \qquad \sigma^2 := \langle (\tfrac{\bar{a}}{a_0} - 1)^2 \rangle$$

Time horizon

Always: $f \in C^2(\mathbb{R} \times \mathbb{R}_+, \mathbb{R})$ with compact support and $T_0 > 0$ arbitrary

Definition (Homogenization time parameter β)

Homogenization works with $\beta \geq 0$ if the solutions u^ε and $\bar{u}$ satisfy

$$\lim_{\varepsilon \to 0} \sup_{t \in [0, T_0 \varepsilon^{-\beta}]} \left\langle \| \partial_t u^\varepsilon(\cdot, t) - \partial_t \bar{u}(\cdot, t) \|_{L^2(\mathbb{R})} \right\rangle = 0$$

$\beta \geq 0$ is the parameter such that homogenization works on $[0, T_0 \varepsilon^{-\beta}]$

Theorem (Main result: Two critical parameters)

1. *For $\beta < \beta_- := \frac{1-\gamma}{1+\gamma}$: For all coefficients (ρ, a) of class γ,* **homogenization works** *with β*
2. *For $\beta > \beta_+ := \frac{1-\gamma}{\gamma}$: There exist coefficients (ρ, a) of class γ such that* **homogenization does not work** *with β*

Examples

$$\beta_- = \frac{1-\gamma}{1+\gamma}, \quad \beta_+ = \frac{1-\gamma}{\gamma}, \quad \text{time interval is } [0, T_0\varepsilon^{-\beta}]$$

- Deterministic medium, $\gamma = 0$: $\beta_- = 1, \quad \beta_+ = \infty$

  Critical for deterministic media is $\beta = 2$

- Very non-ergodic medium, $\gamma = 1$: $\beta_+ = 0$

  Indeed: Homogenization fails for every $\beta > 0$

- Stochastic i.i.d. medium, $\gamma = 1/2$: $\beta_- = 1/3, \quad \beta_+ = 1$

  Theorem, Part 1: For $\beta < 1/3$, classical homogenization is valid

  This matches previous findings: M. Duerinckx, A. Gloria, and M. Ruf.
  A spectral ansatz for the long-time homogenization of the wave equation, 2023

  Theorem, Part 2: Homogenization fails for $\beta > 1$

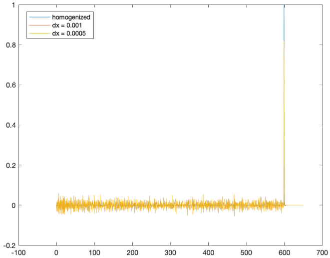
  (for some model with $\gamma = 1/2$)

Open problem

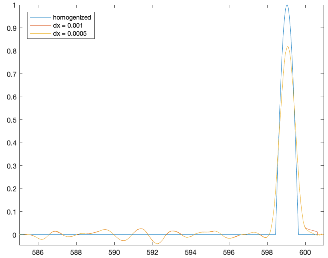
What is the threshold value $\beta_* \in [1/3, 1]$ for i.i.d. media?

Remark (see below): Matched impedance i.i.d medium has $\beta_* = \beta_+ = 1$

Numerical tests



Solutions on $x \in (0, 700)$



Zoom

Numerical results for $\varepsilon = 0.01$ and $t_0 = 600$ and $x_0 = 600$

Calculations with discretization $dx = 0.001$ and with $dx = 0.0005$

Complexity: 10^6 unknowns, time resolution adds a second dimension that requires the same number of points

Transformed wave equation

Diffeomorphism $F : \mathbb{R} \to \mathbb{R}$, new coordinates $z = F(x)$, new function v:

$$v(z) = u(x) \quad \text{for} \quad z = F(x)$$

Transformed coefficients:

$$\tilde{\rho} := (\rho/\partial_x F) \circ F^{-1}\,, \quad \tilde{a} := (a\,\partial_x F) \circ F^{-1}\,, \quad \tilde{f}(z) = (f/\partial_x F) \circ F^{-1}$$

Wave equation for $v(z,t)$

$$\tilde{\rho}(z)\partial_t^2 v(z) - \partial_z(\tilde{a}(z)\partial_z v(z)) = \tilde{f}(z)$$

Proof: The elliptic term of the original equation is

$$\begin{aligned}\partial_x(a(x)\partial_x u(x)) &= \partial_x(a(x)\partial_x(v \circ F)(x)) = \partial_x(a(x)\partial_x F(x)\,\partial_z v(z)) \\ &= \partial_x(\tilde{a}(z)\partial_z v(z)) = \partial_x F\ \partial_z(\tilde{a}(z)\partial_z v(z))\end{aligned}$$

Transformation with $F = \mathrm{id} + \Phi$ implies $\partial_x F = \bar{a}/a(x)$, hence $\tilde{a} = \bar{a}$

$\longrightarrow$ constant coefficient!

On the proofs

Lower bound β_-: Constructing an approximate solution from the homogenized solution, testing procedure

Upper bound β_+: Model class γ, let u^ε be a solution to

$$\rho_\varepsilon \partial_t^2 u^\varepsilon - \partial_x \left(a_\varepsilon \partial_x u^\varepsilon \right) = f$$

Use $F = \mathrm{id} + \Phi$, coordinates $z = F_\varepsilon(x) := \varepsilon F(x/\varepsilon)$ and new function

$$v^\varepsilon(z) = u^\varepsilon(x) \quad \text{for} \quad z = F_\varepsilon(x)$$

Show: Transformed coefficients

$$\tilde{\rho} := (\rho / \partial_y F) \circ F^{-1}, \quad \tilde{a} := (a \, \partial_y F) \circ F^{-1}$$

define again a model of class γ

Assume that both models allow homogenization with parameter β

Contradiction (assuming that homogenization holds for both models)

Growth property of F implies $\rightarrow$ main pulses are at different positions!

Matched impedance

Matched impedance is defined as: $a\rho \equiv 1$
Recall:

$$\tilde{\rho} := (\rho/\partial_x F) \circ F^{-1}\,, \quad \tilde{a} := (a\,\partial_x F) \circ F^{-1}\,,$$

We see: $\partial_x F = 1/a$ is a transformation to constant coefficients

Lemma (d'Alembert formula)

$a, \rho : \mathbb{R} \to [\Lambda^{-1}, \Lambda]$ *bounded with* $a\rho \equiv 1$
Transformation function $F(x) := \int_0^x (a(s))^{-1}\, ds$
For arbitrary $g, h \in L^\infty(\mathbb{R})$ *with compact support, the solution* u *of*

$$\rho \partial_t^2 u - \partial_x(a \partial_x u) = 0 \quad \textit{with} \quad u(\cdot, 0) = g \circ F, \quad \partial_t u(\cdot, 0) = h \circ F$$

is given by

$$u(x,t) = \frac{1}{2}(g(F(x) - t) + g(F(x) + t)) + \frac{1}{2}\int_{F(x)-t}^{F(x)+t} h(y)\, dy$$

Matched impedance corollary

The threshold for i.i.d. coefficients with $a_i\, \rho_i = 1$ is $\beta = \beta_+ = 1$

Conclusions

Problem: Stochastic 1D wave equation with ε-scale

Compare solution u^ε and homogenized solution $\bar{u}$

Question: Does $u^\varepsilon \approx \bar{u}$ hold on time interval $[0, T_0\varepsilon^{-\beta}]$?

- YES for $\beta < \beta_- = \frac{1-\gamma}{1+\gamma}$
- NO for $\beta > \beta_+ = \frac{1-\gamma}{\gamma}$
- Threshold $\beta_* \in [1/3, 1]$ unknown for i.i.d. media
- Threshold is $\beta_* = 1$ for i.i.d. matched impedance media

Thank you!

Reference:

M. Schäffner and B.S. (2024), "The time horizon for stochastic homogenization of the one-dimensional wave equation", Asymptotic Analysis.