# A special perspective on homogenization — with an application to plasticity

## Trends on Applications of Mathematics to Mechanics

## INdAM Workshop, Rome, 2016

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September 8, 2016

# Mathematical analysis of plasticity

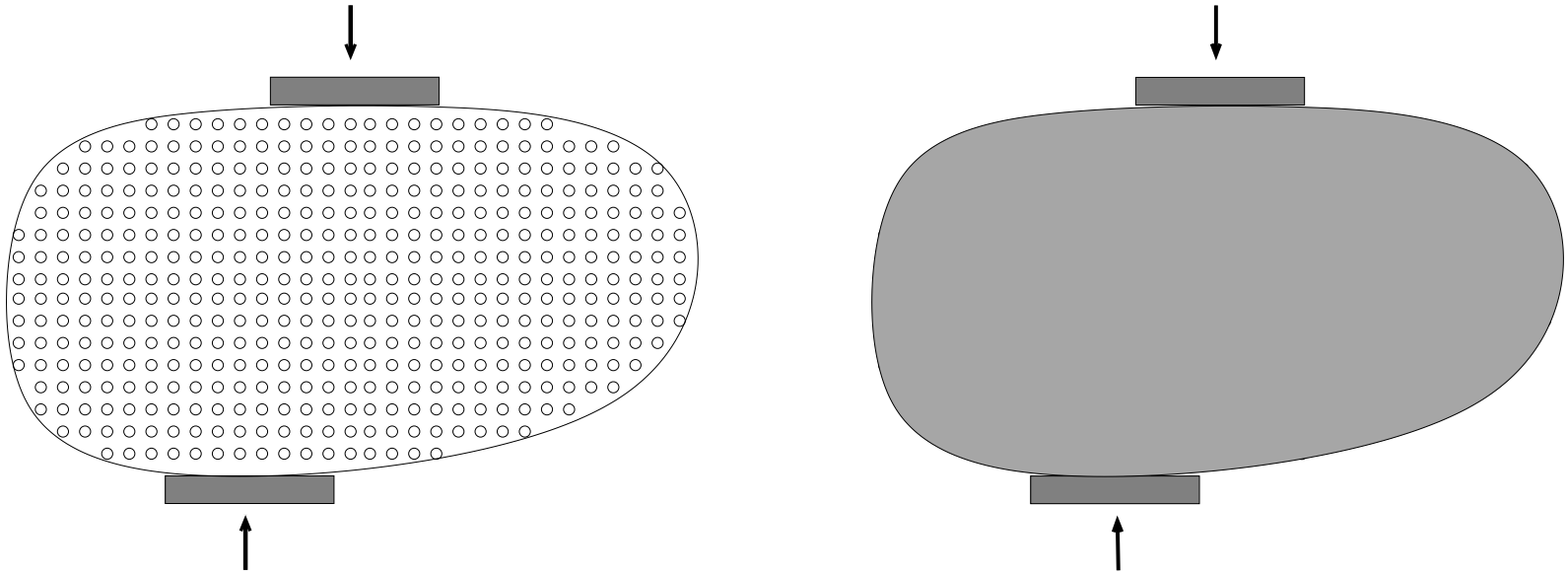
**What can be done mathematically in "plasticity"?**

*(a personal selection)*

1. Analysis of crystal plasticity, derivation of macro-equations
2. Nonlinear theories
3. **Homogenization** (starting from a continuum model)
   - H.-D. Alber(2000)
   - A. Visintin (2005, 2006)
   - A. Mielke & A.M. Timofte / U. Stefanelli / T. Roubicek (2007, 2008, 2013)
   - (H.-D. Alber &) S. Nesenenko (2007, 2009)
   - B.S. & M. Veneroni (2011 & 2014)
   - M. Heida & B.S. (2015 & submitted)

# What is homogenization?

A heterogeneous material occupies a set $\Omega \subset \mathbb{R}^3$. How does it deform under applied forces?

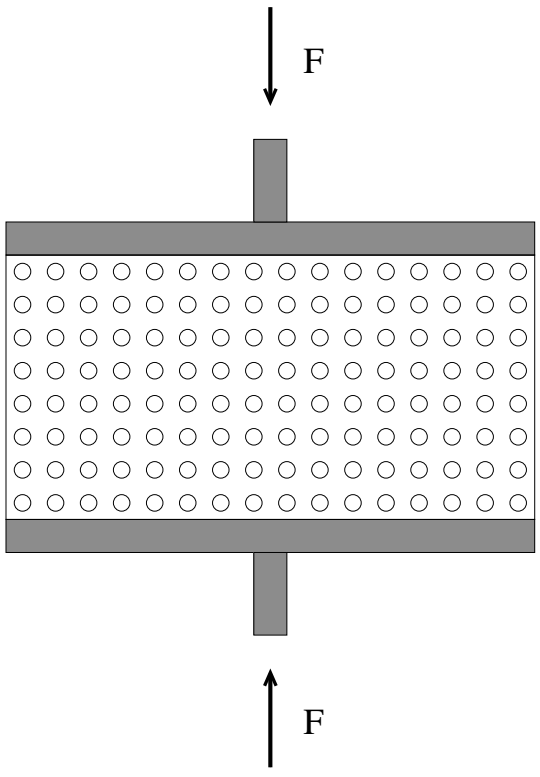


## Aim of homogenization

In good approximation:
the **heterogeneous** material (left)
behaves like the **homogeneous** *effective* material (right).

# What can we measure?

We can make experiments with a heterogeneous test-volume. We deform the specimen (shear and compression) and measure the response




Measurements yield:

- the averaged deformation $\bar{e} \in \mathbb{R}^{n\times n}$ **leads to**
- the averaged force $\bar{\sigma} \in \mathbb{R}^{n\times n}$

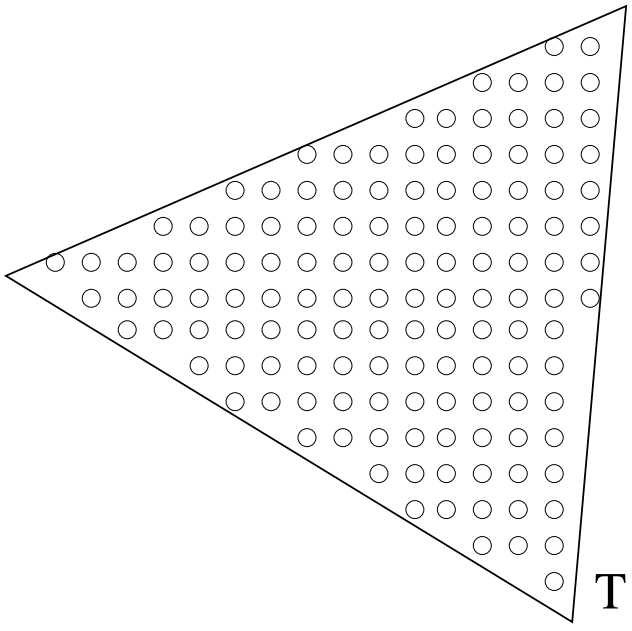
**Result**

We determined the tensor $\mathbf{A}$ in the law

$$\bar{\sigma} = \mathbf{A} \cdot \bar{e}$$

# Effective tensor from deformation experiment

How can we understand the effect of coefficients $a_\varepsilon$?




1. Consider a simplex $\mathcal{T} \subset \mathbb{R}^n$ and a deformation tensor $\xi \in \mathbb{R}^{n\times n}$
2. Impose affine boundary data $U_\xi : \mathbb{R}^n \to \mathbb{R}^n$, $U_\xi(x) := \xi \cdot x$
3. Solve the problem $-\nabla \cdot (a_\varepsilon \nabla u^\varepsilon) = 0$ on $\mathcal{T}$ with $u^\varepsilon = U_\xi$ on $\partial\mathcal{T}$

**Assumption:** The averaged stress is

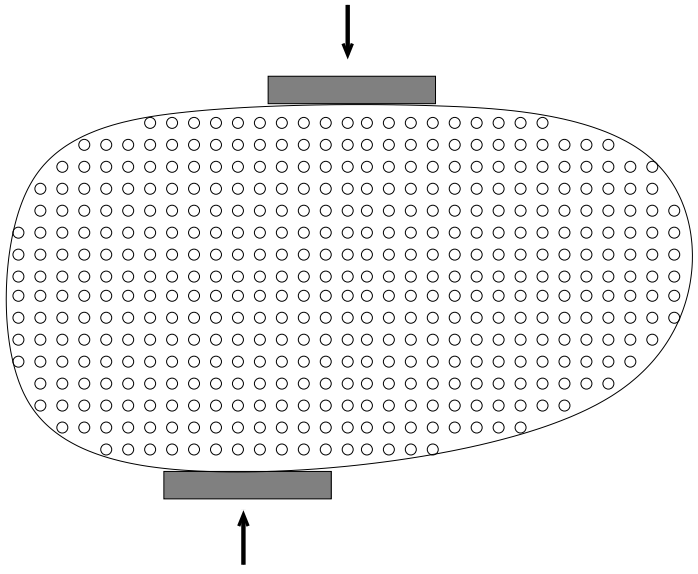
$$\lim_{\varepsilon\to 0} \frac{1}{|\mathcal{T}|} \int_{\mathcal{T}} a_\varepsilon \nabla u^\varepsilon = A \cdot \xi$$

**Under strain $\xi$, the average stress in the test-volume is $A \cdot \xi$**

# Homogenization in more mathematical terms

Heterogeneous material in $\Omega \subset \mathbb{R}^3$. The deformation is

$$u^\varepsilon : \Omega \to \mathbb{R}^3$$



$$\sigma^\varepsilon = a_\varepsilon \nabla u^\varepsilon$$

$$-\nabla \cdot \sigma^\varepsilon = f$$

Effective material description:

$$u^\varepsilon \approx u$$

- deformation $u : \Omega \to \mathbb{R}^3$ with
- stress tensor $\sigma : \Omega \to \mathbb{R}^{3\times 3}$

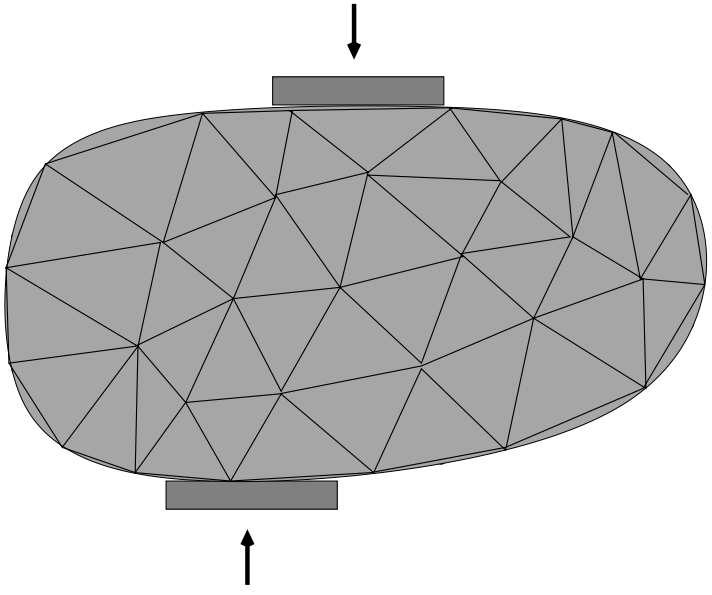
**The aim of homogenization**

The effective law in $\Omega$ is

$$-\nabla \cdot \sigma = f \quad \text{with} \quad \sigma = \mathbf{A} \cdot \nabla u$$

# Coefficients that allow averaging

**In this spirit, non-periodic problems: Needle problem approach!**



**Assumption:** The *single triangle* reacts on an affine deformation $U_\xi$ (with $\nabla U_\xi \equiv \xi$) with the force $A \cdot \xi$

## Definition

The coefficients $a_\varepsilon$ **allow averaging** of the constitutive relation with $A \in \mathbb{R}^{n\times n}$ if, for simplices $\mathcal{T}$, solutions of

$$\begin{aligned} -\nabla \cdot (a_\varepsilon \nabla u^\varepsilon) &= 0 && \text{in } \mathcal{T} \\ u^\varepsilon &= U_\xi && \text{on } \partial\mathcal{T}, \end{aligned}$$

with affine boundary data $U_\xi$ satisfy

$$\lim_{\varepsilon\to 0} \frac{1}{|\mathcal{T}|} \int_{\mathcal{T}} a_\varepsilon \nabla u^\varepsilon = A \cdot \xi$$

# The needle-problem homogenization method

## Theorem (B.S. and M. Veneroni, 2011)

*Assume that the coefficients* $a_\varepsilon$ **allow averaging** *of the constitutive relation with* $A \in \mathbb{R}^{n\times n}$.

**Then:** *Solutions* $u^\varepsilon$ *of* $-\nabla\cdot(a_\varepsilon\nabla u^\varepsilon) = f$ *on domains* $\Omega$ *converge to solutions* $u^*$ *of* $-\nabla\cdot(A\cdot\nabla u^*) = f$.

The proof uses:

- triangulations
- approximate solutions
- adapted grids and a new div-curl Lemma

$$\begin{array}{ccc} u^\varepsilon & \overset{\varepsilon,h\to 0}{\longleftrightarrow} & u_h^\varepsilon \\ & & \big\downarrow \varepsilon\to 0 \\ u^* & \overset{h\to 0}{\longleftarrow} & u_h \end{array}$$

# Definition of the needle problem

**Approximate solutions:** defined as the solutions $u_h^\varepsilon$ of the

## Definition (Needle problem)

Given a Lipschitz domain $\Omega \subset \mathbb{R}^n$, a triangulation $\mathscr{T}_h$ of $\Omega_h \subset \Omega$, and piecewise affine boundary data $\psi$. Function space:

$$\mathcal{N}_h := \left\{ \phi \in H_0^1(\Omega) \,:\, \phi|_{\partial T_k} \text{ is affine for all } T_k \in \mathscr{T}_h,\ \phi \equiv 0 \text{ on } \Omega \setminus \Omega_h \right\}$$

Given $g_h$, the *needle problem* is to find $u_h^\varepsilon \in \psi + \mathcal{N}_h$ such that

$$\int_\Omega a_\varepsilon \nabla u_h^\varepsilon \cdot \nabla \phi = \int_{\Gamma_h} g_h \phi \qquad \forall \phi \in \mathcal{N}_h$$

Here: $f$ is replaced by $g_h$ such that

$$\int_{\bigcup \partial \mathcal{T}_k} g_h \cdot \varphi = \int_\Omega f \cdot \varphi \quad \text{for piecewise affine test functions } \varphi$$

# Proof (homogenization of the elastic problem)

Main part of the homogenization proof:

## Lemma (Comparison of $u_h^\varepsilon$ and $u^\varepsilon$)

*$a_\varepsilon \in L^\infty(\Omega;\mathbb{R}^{n\times n})$ elliptic, $\psi$ boundary values, $u^\varepsilon \in H^1(\Omega)$ solutions of original problem, $u_h^\varepsilon \in \psi + \mathcal{N}_h$ solutions to needle problem. If $\mathscr{T}_h$ are adapted grids for $(u^\varepsilon)_\varepsilon$, then*

$$\lim_{h\to 0}\lim_{\varepsilon\to 0} \|u_h^\varepsilon - u^\varepsilon\|_{H^1(\Omega)} = 0.$$

**Idea of proof:**

1. use $(u^\varepsilon - u_h^\varepsilon)$ as a test-function in the original problem
2. use $(u^\varepsilon - u_h^\varepsilon)$ as a test-function in the needle problem
3. difference yields estimate for $\|u^\varepsilon - u_h^\varepsilon\|^2_{H^1(T)}$
4. show smallness of error terms for $\varepsilon \to 0$ and then $h \to 0$

**the last step uses a div-curl lemma on each triangle**

# Adapted grids

## Theorem (Adapted grids and div-curl lemma)

*Let $\Omega \subset \mathbb{R}^n$, $n = 2$ or $n = 3$ be a bounded Lipschitz domain, $(u^\varepsilon)_\varepsilon$ a bounded sequence in $H^1(\Omega)$.*

1. *To arbitrary $h > 0$ there exists $\Omega_h \subset \Omega$ and an adapted triangulation $\mathscr{T}_h$ of $\Omega_h$ for $(u^\varepsilon)_\varepsilon$.*
2. *Let $(u^\varepsilon)_\varepsilon$ be a sequence with $u^\varepsilon \rightharpoonup u$ weakly in $H^1(\Omega)$ and let $\mathscr{T}_h$ be an adapted grid for $(u^\varepsilon)_\varepsilon$. Let $(q^\varepsilon)_\varepsilon$ be a sequence in $L^2(\Omega, \mathbb{R}^n)$ satisfying*

$$q^\varepsilon \rightharpoonup q \text{ weak in } L^2(\Omega),$$

$$f^\varepsilon := \nabla \cdot q^\varepsilon \to f \text{ strong in } H^{-1}(T), \quad \text{for all } T \in \mathscr{T}_h.$$

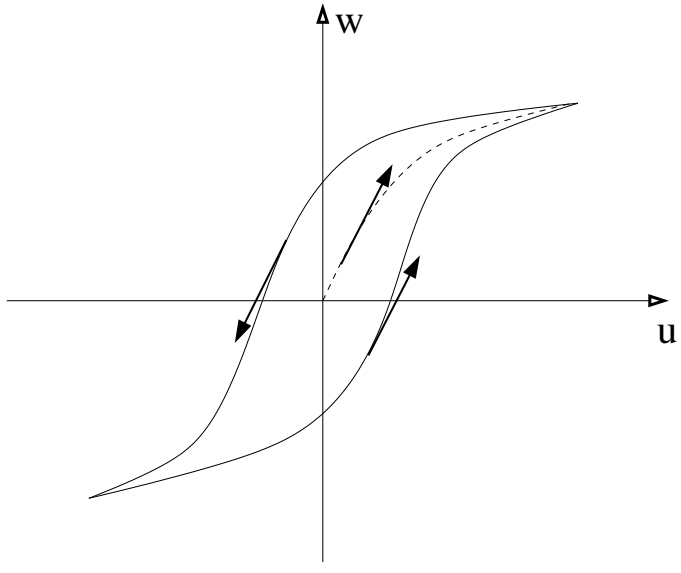
*Then*

$$\lim_{\varepsilon \to 0} \int_{\Omega_h} q^\varepsilon \cdot \nabla u^\varepsilon \, dx = \int_{\Omega_h} q \cdot \nabla u \, dx.$$

**Idea:** boundary values of $u^\varepsilon$ are bounded in $H^1$, hence compact in $H^{1/2}$

# Plasticity as a system with hysteresis

Important in plasticity: **Hysteresis!**




The stress $\sigma(x,t)$ depends not only on the current deformation tensor $\nabla u(x,t)$, but also on its history:

$$\sigma(x,t) = \mathcal{F}\left(\{\nabla u(x,s) | s \in [0,t]\}\right)$$

# Plasticity (Prandtl-Reuss model)

Reference domain of material: $\Omega$, time interval: $(0, T)$

## Variables

| | | | |
|---|---|---|---|
| displacement | $u$ | : | $\Omega \times (0,T) \to \mathbb{R}^n$ |
| strain | $\nabla^s u$ | : | $\Omega \times (0,T) \to \mathbb{R}^{n \times n}$ |
| stress | $\sigma$ | : | $\Omega \times (0,T) \to \mathbb{R}^{n \times n}$ |

with $\nabla^s u = \frac{1}{2}(\nabla u + (\nabla u)^T)$

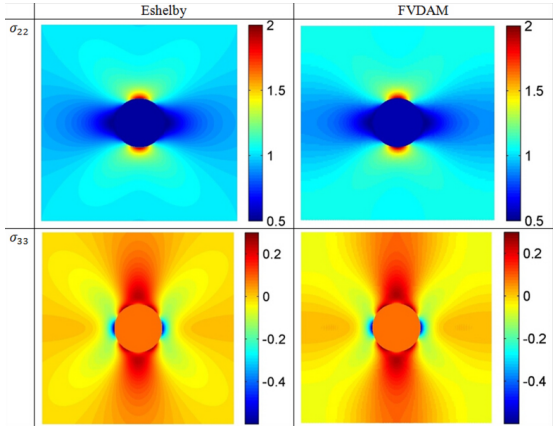
## Equations

| | | |
|---|---|---|
| Conservation of momentum: | $-\nabla \cdot \sigma$ | $= f$ |
| Additive strain decomposition: | $\nabla^s u$ | $= \underbrace{e}_{\text{elastic strain}} + \underbrace{p}_{\text{plastic strain}}$ |
| Hooke's law: | $C\sigma$ | $= e$ |
| Flow rule with kinematic hardening: | $\partial_t p$ | $\in \partial\Psi(\,\sigma - Bp\,)$ |

Given: force $f$, elasticity tensor $C$, hardening tensor $B$, potential $\Psi$

# What can we expect from homogenization?

The periodicity cell remembers its *deformation history*




J. Appl. Mech 81(10), 101005 (Aug 13, 2014)

We cannot expect to find a map

$$\mathbb{R}^{n\times n} \ni \bar{e} \mapsto \bar{\sigma} \in \mathbb{R}^{n\times n}$$

(current local strain)
$\mapsto$ (current local stress)

## Definition (Averaging property in a system with hysteresis)

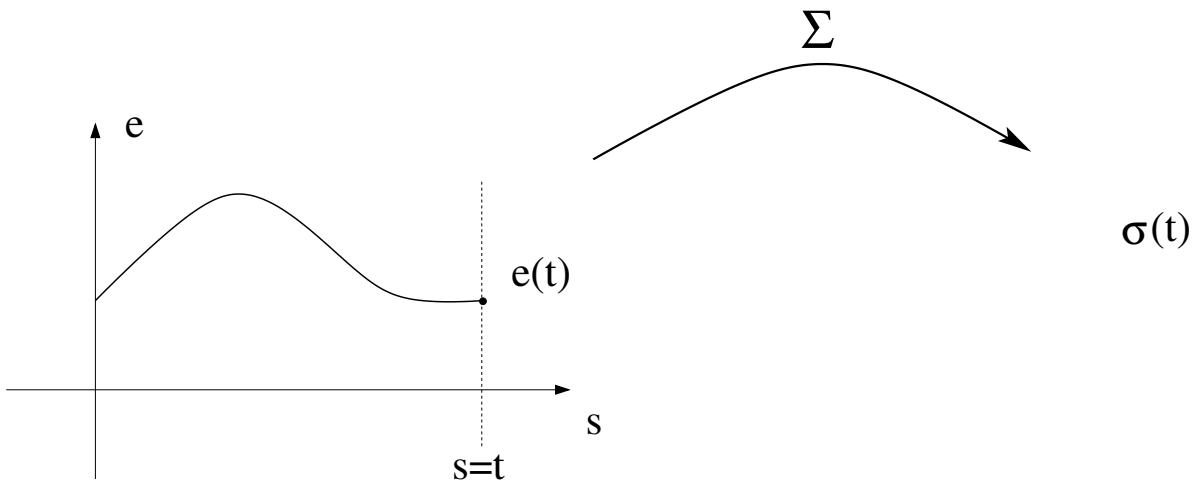
The $\varepsilon$-system allows averaging if there exists an operator

$$\Sigma : H^1(0,T;\mathbb{R}^{n\times n}_s) \to H^1(0,T;\mathbb{R}^{n\times n}_s)$$

such that: For a simplex $\mathcal{T} \subset \mathbb{R}^n$, $\xi \in H^1(0,T;\mathbb{R}^{n\times n}_s)$ and solutions $u^\varepsilon$, $e^\varepsilon$, $p^\varepsilon$, $\sigma^\varepsilon$ to the $\varepsilon$-problem on $\mathcal{T}$ with $f = 0$ and $u^\varepsilon(t)|_{\partial\mathcal{T}} = \xi(t)\cdot x$ holds:

$$\fint_{\mathcal{T}} \sigma^\varepsilon(t) \to \Sigma(\xi)(t)$$

# An averaging property




An evolution of strain is mapped to a stress

## Theorem (Abstract homogenization, M. Heida and B.S.)

Given $\Omega$ and $f$, let the coefficients allow averaging with a lower semi-continuous stress operator $\Sigma$. Then the effective problem

$$-\nabla \cdot \Sigma(\nabla^s u) = f \qquad \text{in } \Omega \times (0,T)$$

has a solution $u$. As $\varepsilon \to 0$, there holds $u^\varepsilon \rightharpoonup u$ in $H^1(0,T;H^1(\Omega))$.

$$\Sigma(\nabla^s u)(x,t) = \Sigma(\nabla^s u(x,.))(t)$$

# Needle problem for plasticity

Discretization of $\Omega_h \subset \Omega$ with grid $\mathbb{T}_h = \{\mathcal{T}_k\}_{k\in\Lambda_h}$
$\mathcal{N}_h := \left\{\phi \in H_0^1(\Omega) : \ \phi|_{\partial\mathcal{T}_k} \text{ is affine } \forall\ k \in \Lambda_h,\ \phi \equiv 0 \text{ on } \Omega \setminus \Omega_h\right\}$
As before: $f$ replaced by $g_h$ such that

$$\int_{\bigcup \partial\mathcal{T}_k} g_h \cdot \varphi = \int_\Omega f \cdot \varphi \quad \text{for piecewise affine } \varphi$$

## Definition (Needle problem in plasticity)

Find $u_h^\varepsilon \in H^1(0,T;\mathcal{N}_h)$, $e_h^\varepsilon, p_h^\varepsilon, \sigma_h^\varepsilon \in H^1(0,T;L^2(\Omega_h;\mathbb{R}_s^{n\times n}))$:

$$\int_0^T \int_{\Omega_h} \sigma_h^\varepsilon : \nabla\varphi = \int_0^T \int_{\bigcup \partial\mathcal{T}_k} g_h \cdot \varphi \qquad \forall \varphi \in L^2(0,T;\mathcal{N}_h)\,,$$

and almost everywhere in $\Omega_h$ holds

$$\nabla^s u_h^\varepsilon = e_h^\varepsilon + p_h^\varepsilon\,, \qquad C_\varepsilon \sigma_h^\varepsilon = e_h^\varepsilon\,, \qquad \partial_t p_h^\varepsilon \in \partial\Psi_\varepsilon\left(\sigma_h^\varepsilon - B_\varepsilon p_h^\varepsilon\right)$$

# Homogenization proof

$u_h^\varepsilon$ = solution of the auxiliary problem, "needle problem"

1. $\lim_{h\to 0}\lim_{\varepsilon\to 0}\|u_h^\varepsilon - u^\varepsilon\|_{L^\infty(0,T;H^1(\Omega))} \to 0$
   - Testing with $\partial_t\left(u_h^\varepsilon - u^\varepsilon\right)$
2. $u_h^\varepsilon \to u_h$ as $\varepsilon \to 0$ where $u_h$ is piecewise affine and solves

$$\int_0^T \int_\Omega \Sigma(\nabla^s u_h) : \nabla\varphi = \int_0^T \int_\Gamma g_h \cdot \varphi \qquad \forall \varphi \in L^2(0,T;Y_h)$$

   $Y_h := \left\{\phi \in H_0^1(\Omega) \,:\, \phi|_{\mathcal{T}_k} \text{ is affine } \forall\, k \in \Lambda_h,\ \phi \equiv 0 \text{ on } \Omega \setminus \Omega_h\right\}$
   - Follows from averaging property
3. $u_h \to u$ as $h \to 0$ where $u$ solves the effective problem
   - Standard finite element calculation

# Stochastic homogenization

Standard setting of stochastic homogenization: Probability space $(\Omega_{\mathcal{P}}, \Sigma_\Omega, \mathcal{P})$, ergodic dynamical system $(\tau_x)_{x\in\mathbb{R}^n}$, the **random coefficients** are

$$C_\varepsilon(x) := C(\tau_{\frac{x}{\varepsilon}}\omega)\,, \quad B_\varepsilon(x) := B(\tau_{\frac{x}{\varepsilon}}\omega)\,, \quad \Psi_\varepsilon(\sigma; x) := \Psi(\sigma; \tau_{\frac{x}{\varepsilon}}\omega)\,.$$

## Theorem (Stochastic homogenization, M. Heida and B.S.)

For $\Sigma : H^1(0,T;\mathbb{R}^{n\times n}_s) \to H^1(0,T;\mathbb{R}^{n\times n}_s)$ from stochastic averaging, the abstract homogenization theorem can be applied:

$$-\nabla\cdot\Sigma(\nabla^s u) = f \qquad \text{in } \Omega\times(0,T)$$

has a solution $u : \Omega\times(0,T)\to\mathbb{R}^n$ and there holds $u^\varepsilon \rightharpoonup u$ as $\varepsilon\to 0$.

$\Sigma$ is given through a cell-problem: Given $\xi : [0,T]\to\mathbb{R}^{n\times n}$, solve

$$\partial_t p(t,\omega) \in \partial\Psi\left(z(t,\omega) - B(\omega)\,p(t,\omega)\,;\,\omega\right)\,, \quad C\,z = \xi + v^s - p\,,$$

with $z(t)\in L^2_{sol}(\Omega_{\mathcal{P}})$, $v(t)\in L^2_{pot}(\Omega_{\mathcal{P}})$, $\qquad \Sigma(\xi) := \int_{\Omega_{\mathcal{P}}} z\,d\mathcal{P}$

# Conclusions

- We presented a new view-point on homogenization
  - needle-problem approach
  - homogenization as two-step procedure
- The method allows to
  - recover known results in elasticity
  - obtain new results in plasticity: **stochastic homogenization**
- Important tool: Adapted grids on which a div-curl lemma holds

Thank you!