Phenomena in the homogenization of Maxwell's equations

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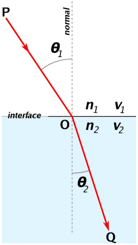
Without Maxwell's Equations

Shortest Paths

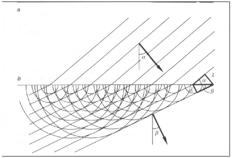
Fermat's principle of **the fastest path:**

Light finds the fastest way to reach the destination,

$$\frac{\sin\Theta_1}{\sin\Theta_2} = \frac{v_1}{v_2} = \frac{n_2}{n_1}$$



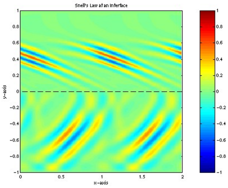
Wave equation



Huygens' principle of **superpositions**

Wave equation

$$\partial_t^2 u = \Delta u$$



Numerical solution

Maxwell's Equations

Variables:

- Electric field E
- Magnetic field H

Simplification:

- Time harmonic solutions

$$H, E \sim e^{-i\omega t}$$

Remarks:

- Vacuum: $\mu = \varepsilon = 1$
- Material parameter

$$\operatorname{Im} \varepsilon \leftrightarrow \text{ conductivity}$$

Maxwell's Equations

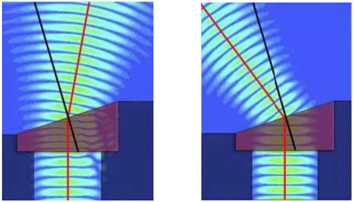
$$\begin{aligned} \operatorname{curl} E &= i\omega\mu H \\ \operatorname{curl} H &= -i\omega\varepsilon E \end{aligned}$$

Negative index of refraction

Veselago (1968)

Properties of materials with negative index, Maxwell equations

If $n_1 > 0$ and $n_2 < 0$, then light should be refracted "backward".

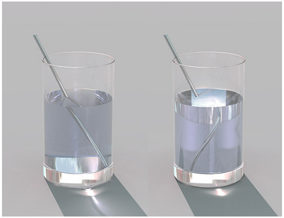


Solutions for positive and negative index

But ... in Maxwell's Equations

- $\mathrm{Re}\,\varepsilon < 0$ possible
- μ is always 1
- $\mathrm{Re}\,\mu\varepsilon < 0$: light can not travel in the medium

Negative Index: ε **and** μ negative!

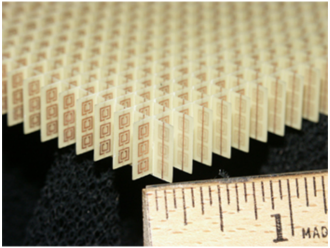


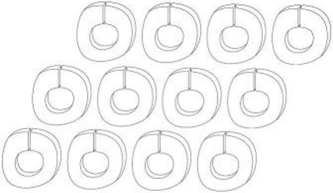
Computer grafics: Negative refraction

Negative index Meta-materials

Wanted: Material with $\operatorname{Re}\mu < 0$

- Pendry et al. ($\sim$ 2000) suggest a split ring construction
- Experiments confirm the negative index





$$\Sigma_\eta := \bigcup_k \eta(k + \Sigma_Y^\eta)$$

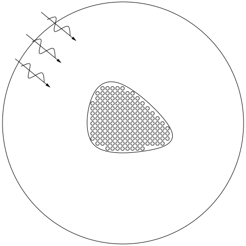
A negative index material in experiments ... and in mathematics

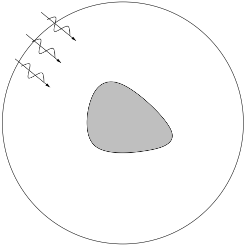
(H_η, E_η) solves the Maxwell system
with a radiation condition

$$\begin{aligned} \operatorname{curl} E_\eta &= \quad i\omega H_\eta \\ \operatorname{curl} H_\eta &= -i\omega\varepsilon_\eta E_\eta \end{aligned}$$

Homogenization

The aim is to replace the **complex structure** of *many* split rings with *high conductivity* by a **homogeneous Meta-material.**





Resulting equations

$$\begin{aligned} \mathrm{curl}\ E &= \quad i\omega\mu_{\mathrm{eff}} H \\ \mathrm{curl}\ H &= -i\omega\varepsilon_{\mathrm{eff}} E \end{aligned}$$

G. Bouchitté and B.S., SIAM J. Mult. Mod. 2010

A. Lamacz and B.S., SIAM J. Math. Anal. 2016

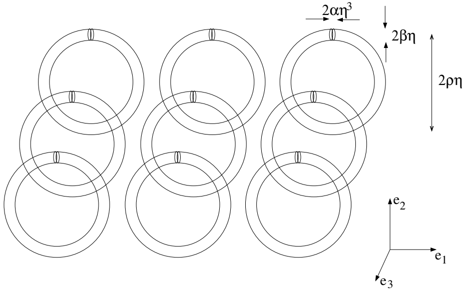
R. Lipton and B.S. Arch. Ration. Mech. Anal. 2018

For appropriate geometry parameters: $\mathrm{Re}(\mu_{\mathrm{eff}}) < 0$

(even though we started from $\mu \equiv 1$)

Microscopic geometry

“Many rings with thin slits”



The material parameter is

$$\varepsilon_\eta = \begin{cases} 1 + i\frac{\kappa}{\eta^2} & \text{in the rings} \\ 1 & \text{else} \end{cases}$$

The parameter η appears $3\times$:

1. many rings
2. high conductivity
3. very thin slit

Formally, 1D-rings

Thin rings:

$$Y = (0,1)^3$$
$$\Sigma = S^1 \subset Y$$
$$\dim(\Sigma) = 1$$

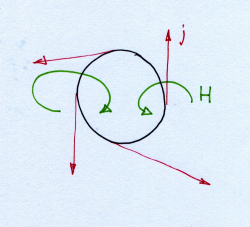
Flux:

$$j_\eta := \eta\varepsilon_\eta E_\eta \rightharpoonup J$$
$$\operatorname{supp} J \subset \Sigma$$
$$\operatorname{div}\ J = 0$$
$$\Rightarrow J = j\tau\mathcal{H}^1 \lfloor \Sigma.$$

$$\begin{aligned} \operatorname{curl}\ E_\eta &= \quad i\omega H_\eta \\ \operatorname{curl}\ H_\eta &= -i\omega\varepsilon_\eta E_\eta \end{aligned}$$

Magnetic field:

$$\operatorname{curl} H^0 = J$$



There is a zero-average cell solution H^0

Its amplitude is given by the limiting flux J in the ring

Homogenization proof

Two-scale convergence: $H_\eta(x) \to H_0(x,y)$ and $E_\eta(x) \to E_0(x,y)$ in the sense of two-scale convergence

Interpretation: In the single periodicity cell $Y = [0,1]^3$ the solution looks like

$$H_\eta(x) \sim H_0(x,y), \qquad E_\eta(x) \sim E_0(x,y)$$

Easy part: The limits $H_0(x,.)$ and $E_0(x,.)$ solve the Maxwell equations in highest order:

$$\begin{aligned} \operatorname{div}_y H_0(x,y) = 0 \text{ in } Y, \quad & \operatorname{curl}_y E_0(x,y) = 0 \text{ in } Y, \\ & \operatorname{curl}_y H_0(x,y) = 0 \text{ in } Y \setminus \Sigma \end{aligned}$$

A new field:

$$J_\eta := \eta \varepsilon_\eta E_\eta \to J_0(x,y)$$

Write the two-scale limit with **four** cell solutions as

$$H_0(x,y) = j(x) H^0(y) + \sum_{k=1}^{3} H_k(x) H^k(y)$$

and determine the **prefactor** j with a slit analysis

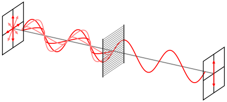
Thin conductive microstructures

Joint with David Wiedemann, TU Dortmund

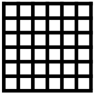
Micro-wave oven

Window shielding

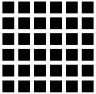
Polarization



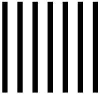
Reflection



Transmission



Polarization



Microwaves: $\lambda \sim 12$ cm

Light: $\lambda \sim 400 - 700$ nm

The η-problem

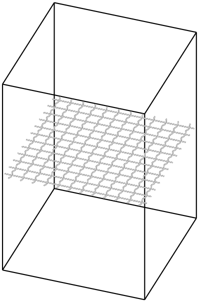
Maxwell's equations

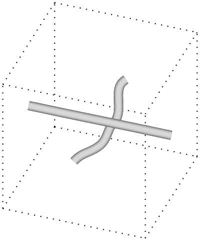
$$\begin{aligned} \mathrm{curl} E^\eta &= i\omega\mu H^\eta + f_h && \text{in } \Omega_\eta \\ \mathrm{curl} H^\eta &= -i\omega\varepsilon E^\eta + f_e && \text{in } \Omega_\eta \\ E^\eta \times \nu &= 0 && \text{on } \partial\Omega_\eta \end{aligned}$$

Homogenization question

Assume $E^\eta \to E^{\text{hom}}$ and $H^\eta \to H^{\text{hom}}$

What equations for E^{hom}, H^{hom}?



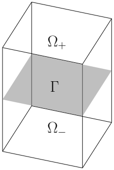


$\Sigma_Y^\eta \subset Y = (0,1)^3$

Macroscopic geometry:

$\Omega = (0,1)^2 \times (-1,1)$

$\Gamma = (0,1)^2 \times \{0\}$



$$\Sigma_\eta = \bigcup_{k \in \mathbb{Z}^2} \eta[(k_1, k_2, 0) + \Sigma_Y^\eta]$$

$$\Omega_\eta := \Omega \setminus \Sigma_\eta$$

Homogenization result

η-problem

$$\begin{aligned} \operatorname{curl} E^\eta &= i\omega\mu H^\eta + f_h && \text{in } \Omega_\eta \\ \operatorname{curl} H^\eta &= -i\omega\varepsilon E^\eta + f_e && \text{in } \Omega_\eta \\ E^\eta \times \nu &= 0 && \text{on } \partial\Omega_\eta \end{aligned}$$

limit problem

$$\begin{aligned} \operatorname{curl} E^{\text{hom}} &= i\omega\mu H^{\text{hom}} + f_h && \text{in } \Omega \\ \operatorname{curl} H^{\text{hom}} &= -i\omega\varepsilon E^{\text{hom}} + f_e && \text{in } \Omega \setminus \Gamma \\ E^{\text{hom}} \times \nu &= 0 && \text{on } \partial\Omega \end{aligned}$$

Theorem (Homogenization result)

Let $(E^\eta, H^\eta) \in L^2(\Omega_\eta, \mathbb{C}^3) \times L^2(\Omega_\eta, \mathbb{C}^3)$ *be solutions of the* η*-problem*

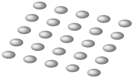
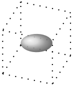
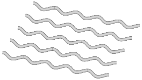
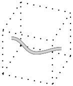
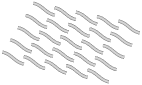
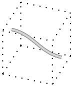
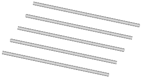
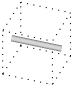
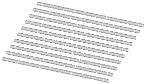
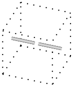
Let trivial extensions converge weakly to (E^{hom}, H^{hom}) *in* $L^2(\Omega, \mathbb{C}^3)$

Then (E^{hom}, H^{hom}) *solves the homogenized system*

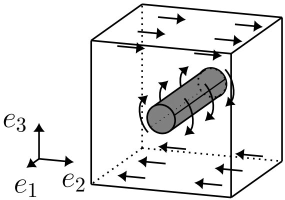
$\longrightarrow$ ***above system together with interface conditions at*** Γ

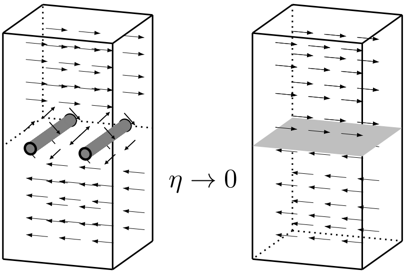
Question: What are the interface conditions?

On (asymptotic) connectedness

obstacle Σ_η	Σ_Y^η	connectedness (asymptotically)
		Disconnected
		Connected
		Disonnected
		Connected (if wires are asymptotically thick)
		Connected if gaps are asymptotically small

Asymptotic connectedness





Definition (Asymptotically connected)

If Ψ_η of H-**type** in direction e_1 exists,

$$\eta^{\frac{1}{2}} \|\Psi_\eta - \operatorname{sgn}(x_3) e_2\|_{L^2(Z^\eta)} \to 0$$

$$\eta^{-\frac{1}{2}} \|\operatorname{curl}\Psi_\eta\|_{L^2(Z^\eta)} \to 0$$

then Σ_Y^η is **asymptotically connected** in direction e_1

Extend cell-function to cylinder

$$Z^\eta = (0,1)^2 \times \mathbb{R} \setminus \Sigma_Y^\eta$$

Connectedness implies vanishing E-field on Γ

Ψ_η a sequence of the H-type (direction e_1), convergences in $L^2(\Omega)$

$$\mathbf{1}_{\Omega_\eta}(\cdot)\Psi_\eta(\cdot/\eta) \to \mathrm{sgn}(x_3)e_2\,, \qquad \mathbf{1}_{\Omega_\eta}(\cdot)(\mathrm{curl}\Psi_\eta)(\cdot/\eta) \to 0$$

A product $\phi_\eta = \varphi(\cdot)\Psi_\eta(\cdot/\eta)$ satisfies

$$\mathbf{1}_{\Omega_\eta}(\cdot)\mathrm{curl}\big(\varphi(\cdot)\Psi_\eta(\cdot/\eta)\big) \to \mathrm{curl}(\varphi e_2)\,\mathrm{sgn}(x_3)$$

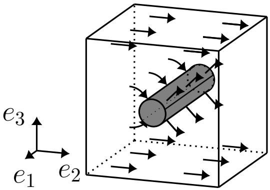
This test-function yields

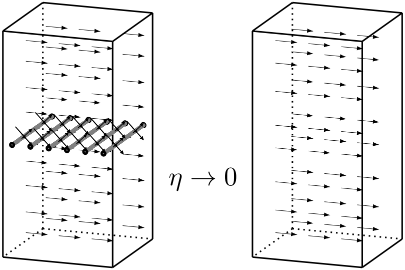
$$\begin{aligned} 0 &= \int_{\Omega_\eta} \mathrm{curl}E^\eta(x)\cdot\phi_\eta(x) - E^\eta(x)\cdot\mathrm{curl}\phi_\eta(x)\,\mathsf{d}x \\ &= \int_\Omega \mathrm{curl}E^\eta\cdot\mathbf{1}_{\Omega_\eta}\varphi\;\Psi_\eta(\cdot/\eta) - E^\eta\cdot\mathbf{1}_{\Omega_\eta}\mathrm{curl}\big(\varphi\Psi_\eta(\cdot/\eta)\big) \\ &\to \int_\Omega \{\mathrm{curl}E\cdot\varphi\,e_2 - E\cdot\mathrm{curl}(\varphi e_2)\}\,\mathrm{sgn}(x_3) \\ &= 2\int_\Gamma (E\times e_2)\cdot e_3\,\varphi = 2\int_\Gamma E_1\,\varphi \end{aligned}$$

Result:

Σ_Y^η asy. connected in e_1
Then $E_1|_\Gamma = 0$

Asymptotic disconnectedness





Definition (Asymptotically disconnected)

If Ψ_η of E-**type** in direction e_2 exists,

$$\Phi_\eta(x) = 0 \quad \text{in } \Sigma_Y^\eta$$

$$\eta^{\frac{1}{2}} \|\Phi_\eta - e_2\|_{L^2(Z)} \to 0$$

$$\eta^{-\frac{1}{2}} \|\mathrm{curl}\Phi_\eta\|_{L^2(Z)} \to 0$$

Σ_Y^η is **asymptotically disconnected**

Result:

Assume: Σ_Y^η is asymptotically disconnected in direction e_2

Then $[\![H_1]\!]_\Gamma = 0$

Limit problem

Theorem (Homogenization result)

$(E^\eta, H^\eta) \in L^2(\Omega_\eta, \mathbb{C}^3) \times L^2(\Omega_\eta, \mathbb{C}^3)$ *solutions to the* η*-problem*
Trivial extensions of E^η *and* H^η *weakly to* (E^{hom}, H^{hom}) *in* $L^2(\Omega)$

Then (E^{hom}, H^{hom}) *solves the following interface conditions:*

Case 1: Reflecting. *If* Σ_Y^η *is asymptotically* ***connected*** *in* e_1 *and* e_2*:*

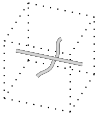
$$E_1^{hom}|_\Gamma = E_2^{hom}|_\Gamma = 0$$

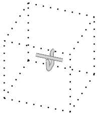
Case 2: Inactive. *If* Σ_Y^η *is asymptotically* ***disconnected*** *in* e_1 *and* e_2*:*

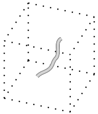
$$[\![H_1^{hom}]\!]_\Gamma = [\![H_2^{hom}]\!]_\Gamma = 0$$

Case 3: Polarising. $j = 3 - i$. *If* Σ_Y^η *is* ***connected*** *in* e_i *and* ***disconnected*** *in* e_j*:*

$$E_i^{hom}|_\Gamma = [\![H_i^{hom}]\!]_\Gamma = 0$$

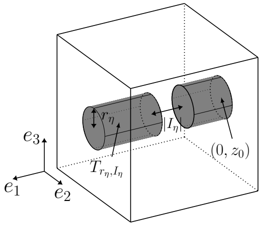


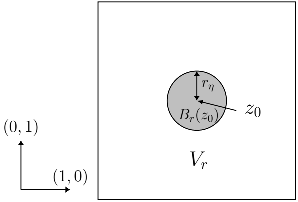




Asymptotic connectedness for wire structures

$$T_{r_\eta} := (0,1) \times B_{r_\eta}(z_0) \qquad T_{r_\eta, I_\eta} := ((0,1) \setminus I_\eta) \times B_{r_\eta}(z_0)$$





Is T_{r_η, I_η} asymptotically connected in the directions e_1 and e_2?

- If $\eta|\ln(r_\eta)| \to 0$ and $\eta^{-1} r_\eta^{-2} |I_\eta| \to 0$, then T_{r_η, I_η} is asymptotically connecting in the direction e_1
- Wires aligned parallel to the e_1 direction are asymptotically disconnected in direction e_2
- If $\eta|\ln(r_\eta)| \to \infty$, then $T_{r_\eta} = T_{r_\eta, \emptyset}$ is asymptotically disconnected in the direction e_1

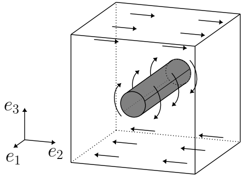
Cell functions of H-type

Definition (Asymptotically connected)

Needs Ψ_η of H-type in e_1:

$$\eta^{\frac{1}{2}} \|\Psi_\eta - \operatorname{sgn}(x_3) e_2\|_{L^2(Z^\eta)} \to 0$$

$$\eta^{-\frac{1}{2}} \|\operatorname{curl}\Psi_\eta\|_{L^2(Z^\eta)} \to 0$$

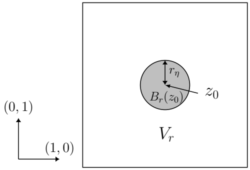


Construction of Ψ

$$\Psi_\eta(x) := \begin{pmatrix} 0 \\ \psi_{r_\eta,1} \\ \psi_{r_\eta,2} \end{pmatrix} (x_2, x_3)\, \mathbf{1}_{\{0<x_3<1\}} + e_2\, \mathbf{1}_{\{x_3>1\}}$$

$$\operatorname{curl}\Psi_\eta(x_3) = \left(\nabla^\perp \cdot \psi_{r_\eta}\right)(x_2, x_3)\, e_1\, \mathbf{1}_{\{0<x_3<1\}}$$

$$\psi_r(z) := \begin{cases} \nabla^\perp u_r^\psi(z) & \text{for } z \in B_r(z_0)\,, \\ \nabla^\perp v_r^\psi(z) & \text{for } z \in V \setminus B_r(z_0) \end{cases}$$



$$\Delta v_r^\psi = 0$$

$$\nabla v_r^\psi \cdot n = \frac{1}{|\partial B_r(z_0)|} \text{ on } \partial B_r\,,$$

$\partial_2 v_r^\psi = -1$ on $\{z_2 = 1\}$, $\quad \partial_2 v_r^\psi = 0$ on $\{z_2 = 0\}$

Need: $\eta |\ln(r_\eta)| \to 0$

Results (for inclusions along a manifold)

- **Homogenized equation depends on connectivity of the inclusions**
- **Connectivity must be understood in an asymptotic sense** It is defined via the existence of cell functions
- **Wire constructions** Wires are connected under the conditions
  - Radius not too small:
    $$\eta |\ln(r_\eta)| \to 0$$
  - Gaps not too wide:
    $$\eta^{-1} r_\eta^{-2} |I_\eta| \to 0$$

Thank you!