Outflow boundary conditions in porous media flow equations

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Why outflow conditions?

- the natural conditions!
- an analytical and numerical challenge

Flow equations

- **Variables** saturation $s : \Omega \to \mathbb{R}$
  pressure $p : \Omega \to \mathbb{R}$
  velocity $v : \Omega \to \mathbb{R}^n$
- **Equations** Darcy-law $v = -k(s)\nabla p$
  mass conservation $\partial_t s + \nabla \cdot v = f$
  capillary pressure $p = p_c(s)$

Together: **Richards' Equation** (neglecting gravity)

$$\partial_t s = \nabla \cdot (k(s)\nabla p_c(s)) + f$$

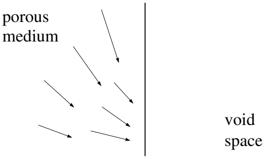
With the Kirchhoff transformation, $\Phi'(s) = k(s)p_c'(s)$:

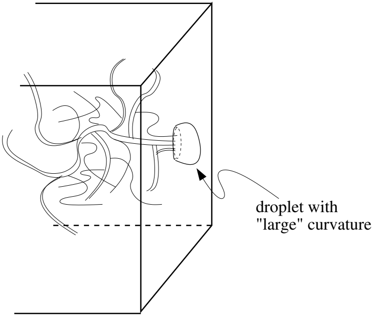
Solve for saturation s and pressure u

$$\partial_t s = \Delta u + f, \qquad u = \Phi(s)$$

Physical description of outflow boundary

We model a porous medium in contact with void space (gas)





- Water can only leave the porous medium: $n \cdot v \geq 0$
- The capillary pressure (=water-pressure) can never exceed 0 — otherwise water exits quickly: $u \leq 0$
- If the capillary pressure is below 0, no water exits: $(n \cdot v)\; u = 0$

($u = 0$ if and only if $p_c(s) = 0$)

Outflow boundary conditions

$u \leq 0$
$n \cdot \nabla u \leq 0$
one is an equality

The condition can be encoded in a weak form as an inequality!

Variational inequality

Demand $u \leq 0$ on Γ_{out} and, for all φ with $\varphi \leq 0$ on Γ_{out},

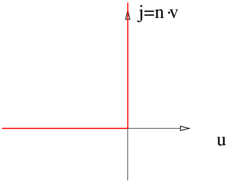
$$\int_{\Omega_T} \partial_t s(\varphi - u) + \nabla u \cdot \nabla(\varphi - u) \geq 0.$$

$(\varphi - u)$ is arbitrary in the interior, hence $\partial_t s = \Delta u$.
Then, formally,

$$\int_{\Gamma_{out} \times (0,T)} n \cdot \nabla u \; (\varphi - u) = \int_{\Omega_T} \Delta u \; (\varphi - u) + \nabla u \cdot \nabla(\varphi - u) \geq 0$$

Hence $n \cdot \nabla u \leq 0$ and $u < 0$ implies $n \cdot \nabla u = 0$ on Γ_{out}.

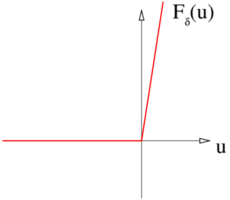
Outflow condition



$$(u, -\partial_n u) \in \mathcal{F}$$

for the Graph $\mathcal{F} \subset \mathbb{R}^2$.

Regularized condition



Regularized outflow condition

$-n \cdot \nabla u_\delta = F_\delta(u_\delta)$ on Γ_{out}

Aim

Solutions (s_δ, u_δ) of

$$\int_{\Omega_T} \{s_\delta\, \partial_t \varphi - \nabla u_\delta \nabla \varphi\} + \int_\Omega s_0 \varphi(0,.) - \int_{\Gamma_{out,T}} F_\delta(u_\delta)\, \varphi = 0 \forall \varphi$$

converge, as $\delta \to 0$, to solutions (s, u) of the outflow problem.

Applications

- Non-degenerate Richards equation
- Degenerate Richards equation
- Two-phase flow

Alt & DiBenedetto Nonsteady flow of water and oil through inhomogeneous porous media. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 12(3):335–392, 1985.

Alt, Luckhaus, Visintin On nonstationary flow through porous media. *Ann. Mat. Pura Appl. (4)*, 136:303–316, 1984.

Arbogast The existence of weak solutions to single porosity and simple dual-porosity models of two-phase incompressible flow. *Nonlinear Anal.*, 19(11):1009–1031, 1992.

Kröner & Luckhaus Flow of oil and water in a porous medium. *J. Differential Equations*, 55(2):276–288, 1984.

Pop & S. *Regularization schemes for non-degenerate Richards equations and outflow conditions.* (in preparation).

S. *Regularization of outflow problems in unsaturated porous media with dry regions.* J. Differential Equations 237:278-306, 2007.

Lenzinger & S. *Two-phase flow equations with outflow boundary conditions in the hydrophobic-hydrophilic case.* (Preprint 2008, submitted)

Ohlberger & S. *Modelling of interfaces in unsaturated porous media.* Conference Proceedings of the AIMS, 2008.

Definition (Variational solution of the limit problem)

$(s,u) \in L^2(\Omega_T) \times L^2(\Omega_T)$ with $u = \Phi(s)$ a.e. is a *variational solution*, if $\partial_t s \in L^2(\Omega_T)$ with $s(0) = s_0$, $\nabla u \in L^2(\Omega_T)$, $u = 0$ on Γ_D, $u \leq 0$ on Γ_{out}, and

$$\int_{\Omega_T} \partial_t s H(\varphi - u) + \nabla u \cdot \nabla[H(\varphi - u)] \geq 0$$

for all $\varphi \in L^2(0,T;H^1(\Omega))$ with $\varphi = 0$ on Γ_D and $\varphi \leq 0$ on Γ_{out}, and all $H : \mathbb{R} \to \mathbb{R}$ of class C_b^1, monotonically increasing with $H(0) = 0$.

Theorem

There exists a unique variational solution.

Uniqueness: We use $H = \mathrm{sign}$.
Existence: The equality for u_δ reads

$$\int_{\Omega_T} \partial_t s_\delta H(\varphi - u_\delta) + \nabla u_\delta \nabla H(\varphi - u_\delta)$$
$$= - \int_{(0,T)\times\Gamma_{out}} F_\delta(u_\delta)\, H(\varphi - u_\delta) \geq 0.$$

The last inequality by distinguishing two cases.
For the limit $\delta \longrightarrow 0$ we use

$$\limsup_{\delta \to 0} \int_{\Omega_T} \nabla u_\delta \nabla H(\varphi - u_\delta) \leq \int_{\Omega_T} \nabla u \nabla H(\varphi - u).$$

Conclusions

- non-degenerate Φ implies strong convergences
- regularized outflow condition relates to energy loss

$\longrightarrow$ **Existence theorem**

Degenerate case. Under appropriate assumptions ...

Theorem (B. S. 2007)

$(s_\delta, u_\delta) \to (s, u)$ weakly in $L^\infty(\Omega_T) \times L^2(0,T;H^1(\Omega))$. With $v = -\nabla u$

$$\partial_t s + \mathrm{div}\ v = 0 \text{ in } \mathcal{D}'(\Omega_T)$$

and $u \in \Phi(s)$ a.e. in Ω_T. On the outflow boundary, as distributions,

$$v \cdot n \geq 0, \qquad K(s) - K(a_0) \leq 0$$
$$(v \cdot n) \cdot (K(s) - K(a_0)) \geq 0.$$

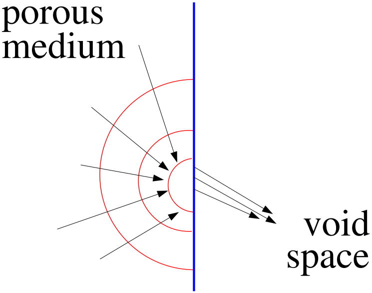
Note: $p_c(a_0) = 0$, the function $s \mapsto K(s) := k^2(s)s$ is monotone.

On the proof

Lemma (Defect measure)

For a measure $\nu \in \mathcal{M}(\Gamma_{out,T})$ with $\nu \geq 0$, for $\delta \to 0$,

$$(K(s_\delta)\, v_\delta \cdot n)|_{\Gamma_{out,T}} \rightharpoonup (K(s)\, v \cdot n)|_{\Gamma_{out,T}} - \nu \text{ in } \mathcal{D}'(\Gamma_{out,T}).$$



In the bulk term

$$\nabla[K(s_\delta)] \cdot v_\delta = (\partial_s K(s_\delta)\, \nabla s_\delta + \nabla_x K(s_\delta)) \cdot$$
$$\cdot(-k_\delta(s_\delta)\partial_s \rho_\delta(s_\delta) \nabla s_\delta - k_\delta(s_\delta)\nabla_x \rho_\delta(s_\delta))$$

second terms converge strongly in $L^2(\Omega_T)$

the singular part is generated by

$-\partial_s K(s_\delta)\, k_\delta(s_\delta)\partial_s \rho_\delta(s_\delta) |\nabla s_\delta|^2 \leq 0$

Two-phase flow

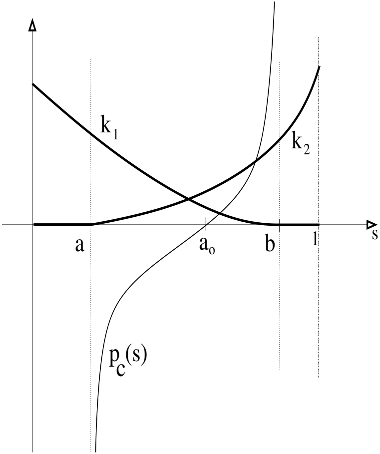
Both phases (liquid and gas) satisfy **Darcy's law** and **mass-conservation.**

Two-phase flow

$$\partial_t s_1 = \nabla \cdot (k_1(s_1)\nabla p_1) + f_1$$
$$\partial_t s_2 = \nabla \cdot (k_2(s_1)\nabla p_2) + f_2$$
$$s_1 + s_2 = 1$$
$$p_1 - p_2 = p_c(s_1)$$

Outflow condition

$$p_2 = 0$$
$$p_1 \leq 0$$
$$-\partial_n p_1 \geq 0$$
$$p_1 < 0 \Rightarrow \partial_n p_1 = 0$$



Global pressure. Define

$$p = p_2 + \int_{\bar{s}}^{s} \frac{k_1}{k_1 + k_2} p_c'$$

Why using the global pressure?

p satisfies an equation of the kind $\operatorname{div}(k \nabla p) = 0$.

- *maximum principle.* p has global maximum at boundary
- *regularity estimates.* p as regular as boundary values

Problem

We do not have a **boundary condition** for p!

Lemma

Under appropriate assumptions ...

The saturation remains bounded away from critical values.

Maximum principle

- In an inner maximum of s:
  - $\partial_t s \geq 0$ implies $\Delta p_1 \geq 0$ and $\Delta p_2 \leq 0$
  - $p_c(s)$ has a maximum, hence $\Delta[p_c(s)] \leq 0$
- In a maximum of s at outflow boundary:
  - geometric condition: $n \cdot \nabla s \geq 0$ and $n \cdot \nabla p_c(s) \geq 0$
  - By $p_2 = 0$, the point is simultaneously maximum of the global pressure, hence $n \cdot \nabla p \geq 0$
  - By $n \cdot \nabla p_1 \sim n \cdot \nabla p + n \cdot \nabla p_c \geq 0$: no outflow, $p_1 = 0$.

Lemma

The saturation remains bounded away from critical values.

For a proof:

1. Regularize outflow condition
2. Discretize in time

Under appropriate assumptions ...

Theorem (M. Lenzinger and B. S. 2008)

$(s^h, p_1^h, p_2^h) \to (s, p_1, p_2)$ for $(\delta, h) \to 0$. The limit satisfies

$$\begin{aligned} \partial_t s - \nabla \cdot (k_1(s)\, \nabla p_1) &= 0 \text{ in } \mathcal{D}'(\Omega_T), \\ -\partial_t s - \nabla \cdot (k_1(s)\, \nabla p_2) &= 0 \text{ in } \mathcal{D}'(\Omega_T), \\ p_1 - p_2 &= p_c(s(.)) \text{ a.e. in } \Omega_T. \end{aligned}$$

At the outflow boundary we have $p_2 = 0$, $p_1 \leq 0$ in the sense of traces and $v_1 \cdot n \geq 0$ in the distributional sense. For a.e. $t \in (0,T)$ holds

$$- \int_\Omega (P_c(s(t)) - P_c(s^0)) + \int_\Omega s(t)(\phi_1 - \phi_2)(t)\Big|_0^t - \int_{\Omega_t} s\; \partial_t(\phi_1 - \phi_2) + \sum_j \int_{\Omega_t} k_j(s)\, \nabla p_j\, \nabla(\phi_j - p_j) \geq 0$$

for all $\phi_j \in C^1(\overline{\Omega}_T)$, $\phi_j = p_j^D$ on Γ^D, $\phi_1 \leq 0$ and $\phi_2 = 0$ on Γ_{out}.

Conclusions

- Most natural boundary conditions: “Neumann” and “Outflow”
- The outflow be aproximated well with the

Regularized outflow condition

$$-n \cdot \nabla u_\delta = F_\delta(u_\delta) \text{ on } \Gamma_{out}$$

- Rigorous results for Richards and two-phase flow

Open problems in degenerate cases

- uniqueness in degenerate Richards
- degenerate regions in two-phase flow

Thank You!