

Time harmonic Maxwell's equations in periodic waveguides

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Andreas Kirsch, Ben Schweizer

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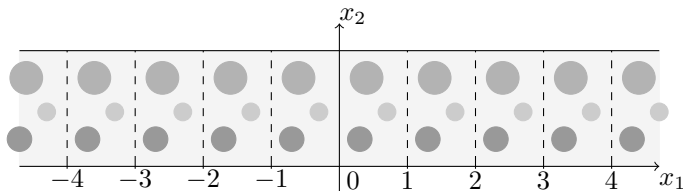
Maxwell's equations in a wave-guide

Given: Frequency $\omega > 0$, permeability $\mu = \mu(x)$, permittivity $\varepsilon = \varepsilon(x)$, a right hand side (f_h, f_e)

Goal: Solve **Maxwell's equations**

$$\begin{aligned}\operatorname{curl} E &= i\omega\mu H + f_h \\ \operatorname{curl} H &= -i\omega\varepsilon E + f_e\end{aligned}\tag{1}$$

in a **waveguide geometry:** $\Omega = \mathbb{R} \times S \subset \mathbb{R}^3$
with $S \subset \mathbb{R}^2$ a bounded Lipschitz domain



The simplest related equation

An ordinary differential equation on \mathbb{R}

Given $f : \mathbb{R} \rightarrow \mathbb{C}$ with support in $[-R, R]$ and $\omega > 0$,
find $u : \mathbb{R} \rightarrow \mathbb{C}$ solving

$$\partial_x^2 u + \omega^2 u = f$$

Solution: For some $a_1, a_2, b_1, b_2 \in \mathbb{C}$:

For $x > R$, the solution is $u(x) = a_1 e^{i\omega x} + a_2 e^{-i\omega x}$

For $x < -R$, the solution is $u(x) = b_1 e^{i\omega x} + b_2 e^{-i\omega x}$

Radiation condition: One might want, e.g., $a_2 = b_1 = 0$

Robin conditions in $x = R$ and $x = -R$ provide the solution u

Note: In general, $u \notin H^1(\mathbb{R}, \mathbb{C})$

Reformulation of Maxwell's equations

Goal: For $\omega > 0$, $\mu = \mu(x)$, $\varepsilon = \varepsilon(x)$, $(f_h, f_e) = (f_h, f_e)(x)$, solve

$$\operatorname{curl} E = i\omega\mu H + f_h$$

$$\operatorname{curl} H = -i\omega\varepsilon E + f_e$$

with the boundary conditions for perfect conductors: $E \times \nu = 0$ on $\partial\Omega$

Periodicity, positivity: $\varepsilon, \mu \in L^\infty(\Omega)$ strictly positive, 2π -periodic in x_1

Decay property of the right hand side: $\int_\Omega (1 + x_1^2)^2 |f(x)|^2 dx < \infty$

Weak formulation with $u := H$ as the only unknown

$$\int_\Omega \left\{ \frac{1}{\varepsilon} \operatorname{curl} u \cdot \operatorname{curl} \bar{\phi} - \omega^2 \mu u \bar{\phi} \right\} = \int_\Omega \left\{ \frac{1}{\varepsilon} f_e \operatorname{curl} \bar{\phi} - i\omega f_h \bar{\phi} \right\} \quad (2)$$

for every $\phi \in H^1(\Omega, \mathbb{C}^3)$ with bounded support

This already encodes the boundary conditions

Useful function space:

$$u \in H(\operatorname{curl}, \Omega) := \{u \in L^2(\Omega, \mathbb{C}^3) \mid \operatorname{curl} u \in L^2(\Omega, \mathbb{C}^3)\}$$

Floquet-Bloch transformation

Periodicity cell: $W := (0, 2\pi) \times S$ for $\Omega = \mathbb{R} \times S$

Quasimoments: $\alpha \in I := [-1/2, 1/2]$

Floquet-Bloch transform

$$\mathcal{F}_{\text{FB}} : L^2(\Omega) \rightarrow L^2(W \times I), \quad u = u(x) \mapsto \hat{u} = \hat{u}(x, \alpha)$$

For smooth functions u with compact support, $x = (x_1, \tilde{x})$:

$$\hat{u}((x_1, \tilde{x}), \alpha) := \sum_{\ell \in \mathbb{Z}} u(x_1 + 2\pi\ell, \tilde{x}) e^{-i\ell 2\pi\alpha}$$

Inverse: For arbitrary $y \in \Omega$, reconstruct with

$$u(y) = \int_I \hat{u}(y, \alpha) d\alpha$$

With these formulas also: \mathcal{F}_{FB} is an isomorphism with bounded inverse

$$\mathcal{F}_{\text{FB}}^{-1} : L^2(I, H_\alpha^1(W)) \rightarrow H^1(\Omega)$$

Floquet-Bloch transformed equation

$$W := (0, 2\pi) \times S, \quad \alpha \in I := [-1/2, 1/2]$$

Function space: $X := H_{\text{per}}(\text{curl}, W)$ with scalar product

$$\langle u, \phi \rangle_X := \left\langle \frac{1}{\varepsilon} \text{curl } u, \text{curl } \phi \right\rangle_{L^2(W)} + \langle \mu u, \phi \rangle_{L^2(W)}$$

The Maxwell operator

For every $\alpha \in I$, a linear operator $L_\alpha : X \rightarrow X$ is defined by

$$\langle L_\alpha v, \varphi \rangle_X := \int_W \frac{1}{\varepsilon} \text{curl}(v e^{i\alpha x_1}) \cdot \text{curl}(\overline{\varphi e^{i\alpha x_1}}) - \omega^2 \mu v \cdot \bar{\varphi}$$

The right hand side (f_h, f_e) is represented with a family $y_\alpha \in X$

Equivalent formulation

Maxwell is solved with $u \in H^1(\Omega)$ when we show: For almost every $\alpha \in I$, there is $v(\cdot, \alpha) \in X$ solving

$$L_\alpha v(\cdot, \alpha) = y_\alpha \tag{3}$$

and there holds $v \in L^2(I, X)$

Critical α -values

Trivial case: When L_α^{-1} exists for all α , then $v(\cdot, \alpha) = L_\alpha^{-1}(y_\alpha)$

Critical α -values

For $\alpha \in I$ let Y^α be the space of α -quasiperiodic solutions to the homogeneous problem. Critical values:

$$\mathcal{A} := \{\alpha \in [-1/2, 1/2] \mid Y^\alpha \neq \{0\}\}$$

$\longrightarrow Y^\alpha$ consists of propagating modes

Energy transport is encoded with an hermitean form:

$$Q(u, \phi) := i \int_W \frac{1}{\varepsilon} [(\operatorname{curl} u \times \bar{\phi}) - (\operatorname{curl} \bar{\phi} \times u)] \cdot e_1$$

u a propagating mode

u transports energy to the right $\iff Q(u, u) > 0$

u transports energy to the left $\iff Q(u, u) < 0$

Assumption

For every $0 \neq \phi \in Y^\alpha$, the map $Q(\cdot, \phi) : Y^\alpha \rightarrow \mathbb{C}$ is non-trivial

Functional analysis

Definition (Regular C^1 -family)

$(L_\alpha)_\alpha$ is a regular C^1 -family when:

1. L_α is a self-adjoint Fredholm operator with index 0 (for every α)
2. The operators depend differentiable on α
3. The derivatives $\partial_\alpha L_\alpha$ are invertible on the kernel for every α

Theorem (Functional analysis, Kirsch et al.)

Let $(L_\alpha)_\alpha$ be a regular C^1 -family of operators

Let $\alpha \mapsto y_\alpha \in L_\alpha(X)$ map into the image and be Lipschitz

Then $v(\cdot, \alpha) = L_\alpha^{-1}(y_\alpha)$ is uniformly bounded

Proof with implicit function theorem

Information regarding Maxwell: $(L_\alpha)_\alpha$ is a regular C^1 -family

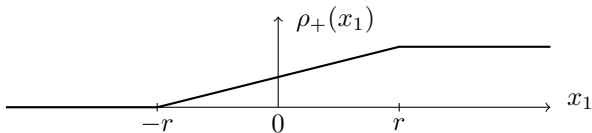
Relevant step: Fredholm property of L_α

Invertibility of $\partial_\alpha L_\alpha$ on the kernel follows from Q -assumption
(Loosely speaking: Q is the derivative $\partial_\alpha L_\alpha$)

Problem: In general, $y_\alpha \notin L_\alpha(X)$

Indeed: When $y_\alpha \in L_\alpha(X)$ for all α , we find $u \in L^2(\Omega)$!

Cut-off functions: ρ_+ with limits 1 and 0 and $\rho_- := 1 - \rho_+$



Let $(\phi_\ell)_\ell$ be the quasiperiodic homogeneous solutions
For every ℓ , let ρ_ℓ be one of the two cut-off functions

Definition (Propagating part and radiation condition)

(i) *Propagating part.* For complex coefficients $(a_\ell)_{1 \leq \ell \leq L}$,

$$u^{\text{prop}} := \sum_{\ell=1}^L a_\ell \rho_\ell \phi_\ell$$

is the propagating wave function corresponding to $a \in \mathbb{C}^L$

(ii) *Radiation condition.* A solution $u \in H_{\text{loc}}(\text{curl}, \Omega)$ of (2) satisfies the radiation condition, when there exists $a \in \mathbb{C}^L$ such that

$$u^{\text{rad}} := u - u^{\text{prop}} \in H(\text{curl}, \Omega)$$

Existence result

Theorem (Existence and uniqueness of solutions to the radiation problem)

Let S , ω , ε , μ , f_e and f_h be given, let the Q -assumption be satisfied. Then (2) has a unique solution $u \in H_{\text{loc}}(\text{curl}, \Omega)$ satisfying the radiation condition. With $C = C(S, \varepsilon, \mu, \omega, \rho_{\pm})$ holds

$$\|u^{\text{rad}}\|_{H(\text{curl}, \Omega)} + \|u^{\text{prop}}|_W\|_{H(\text{curl}, W)} \leq C (\|f_e\|_{L_*^2(\Omega)} + \|f_h\|_{L_*^2(\Omega)})$$

The coefficients $(a_\ell)_\ell$ are given by

$$a_\ell = \frac{2\pi i}{|Q(\phi_\ell, \phi_\ell)|} (\langle \varepsilon^{-1} f_e, \text{curl} \phi_\ell \rangle_{L^2(\Omega)} - \langle i\omega f_h, \phi_\ell \rangle_{L^2(\Omega)})$$

Sketch of proof

Step 1: Under some orthogonality condition holds $y_\alpha \in L_\alpha(X)$ for all α . Functional analysis theorem yields the solution in $u \in H^1(\Omega)$.

Step 2: General case. Pre-factors $(a_\ell)_\ell \in \mathbb{C}^L$ yield u^{prop} . $u - u^{\text{prop}}$ satisfies a different problem. Show that $a_\ell \in \mathbb{C}$ can be chosen such that the problem for $u - u^{\text{prop}}$ satisfies the orthogonality condition.

Fredholm property

Here: $\alpha \in I$ is fixed, any dependence on α is suppressed

$W = (0, 2\pi) \times S$, $\varepsilon, \mu \in L^\infty(\Omega)$ real valued and positive

Function space: $H_\alpha(\text{curl}, W)$ with scalar product

$$\langle u, \varphi \rangle_{H(\text{curl}, W)} := \int_W \left\{ \frac{1}{\varepsilon} \text{curl } u \cdot \text{curl } \bar{\varphi} + \mu u \cdot \bar{\varphi} \right\}$$

Operator: $L : H_\alpha(\text{curl}, W) \rightarrow H_\alpha(\text{curl}, W)$ defined by

$$\langle Lu, \varphi \rangle_{H(\text{curl}, W)} = \int_W \left\{ \frac{1}{\varepsilon} \text{curl } u \cdot \text{curl } \bar{\varphi} - \omega^2 \mu u \cdot \bar{\varphi} \right\}$$

Helmholtz decomposition: $H_\alpha(\text{curl}, W) = D \oplus G$:

$$D := \left\{ u \in H_\alpha(\text{curl}, W) \mid \int_W \mu u \cdot \nabla \psi = 0 \text{ for all } \psi \in H_\alpha^1(W) \right\}$$

$$G := \{ v \in H_\alpha(\text{curl}, W) \mid \exists \psi \in H_\alpha^1(W) : v = \nabla \psi \}$$

$\longrightarrow H(\text{curl}, W)$ -orthogonal complements

Fredholm property

Lemma (Fredholm property)

The operator L is a self-adjoint Fredholm operator with index 0.

Consider $v \in G$ and $Lv \in X$ and $u \in D$:

$$\langle Lv, u \rangle_{H(\text{curl}, W)} = -\omega^2 \langle \mu v, u \rangle_{L^2(W)} = -\omega^2 \langle v, \mu u \rangle_{L^2(W)} = 0$$

This provides $L|_G : G \rightarrow G$. Similarly: $L|_D : D \rightarrow D$.

Hence, on $H_\alpha(\text{curl}, W) = D \oplus G$:

$$L = \begin{pmatrix} L|_D & 0 \\ 0 & L|_G \end{pmatrix}$$

$L|_D : D \rightarrow D$ is a Fredholm operator with index 0: One shows that $K := L - \text{id}$ is a compact operator $D \rightarrow D$.

On G , the operator L is nothing but multiplication with $-\omega^2$, hence a Fredholm operator with index 0.

$$Y = B$$

Two function spaces of modes

The span of quasiperiodic solutions

$$Y := \bigoplus_{j=1}^J Y_j \subset H(\text{curl}, W), \quad \text{identified with } Y \subset H_{\text{loc}}(\text{curl}, \Omega)$$

The space of bounded solutions, $\|U\|_{sL} := \sup_{r \in 2\pi\mathbb{Z}} \|U|_{W_r}\|_{L^2(W_r)}$:

$$B := \{U \in H_{\text{loc}}(\text{curl}, \Omega) \mid U \text{ solves (2) for } f_e = f_h = 0, \|U\|_{sL} < \infty\}$$

Theorem (Characterization of bounded homogeneous solutions)

When the Q -assumption is satisfied, then

$$Y = B$$

Proof of $Y = B$.

We consider $U \in B$ and want to show $U \in Y$

Let $f = f_h$ with compact support be arbitrary

For f , let the Maxwell solution be $u = u^{\text{prop}} + u^{\text{rad}}$ with $(a_\ell)_{1 \leq \ell \leq L}$

With cut-off function ϑ_R , use $U\vartheta_R$ as a test-function

$$\int_{\Omega} \left\{ \frac{1}{\varepsilon} \operatorname{curl} u \cdot \operatorname{curl} (\bar{U} \vartheta_R) - \omega^2 \mu u \cdot \bar{U} \vartheta_R \right\} = -i\omega \int_{\Omega} f \cdot \bar{U}$$

Evaluate the left hand side using that U is a homogeneous solution:

With c_ℓ depending on U and ϕ_ℓ , but not on f , we find

$$\sum_{\ell=1}^L c_\ell a_\ell = -i\omega \langle f, U \rangle_{L^2(\Omega)}$$

We now recall: a_ℓ is a linear combination of $\langle f, \phi_k \rangle_{L^2(\Omega)}$.

Result, since f was arbitrary: U is a linear combination of the ϕ_k .

Locally perturbed media

Theorem (Fredholm alternative for perturbed media)

Let $\mu_{\text{per}}, \varepsilon_{\text{per}} \in L^\infty(\Omega)$ be periodic functions with positive lower bounds

Let $\mu, \varepsilon \in L^\infty(\Omega)$ be given as compact perturbations of $\mu_{\text{per}}, \varepsilon_{\text{per}}$

We assume positive lower bounds also for μ, ε

Let the Q -assumption be satisfied for $\mu_{\text{per}}, \varepsilon_{\text{per}}$

Let $u = 0$ be the only solution to the homogeneous perturbed system

Then there exists a unique radiating solution for every (f_e, f_h)

Hint on the proof: With operators $D : X \rightarrow Y$ and $\xi, Q : Y \rightarrow Y$

$$D := \begin{pmatrix} 0 & \text{curl} \\ -\text{curl} & 0 \end{pmatrix}, \quad \xi := \begin{pmatrix} \varepsilon_{\text{per}} & 0 \\ 0 & \mu_{\text{per}} \end{pmatrix}, \quad Q := \begin{pmatrix} q_\varepsilon & 0 \\ 0 & q_\mu \end{pmatrix}$$

Maxwell's equations take the form

$$(D + i\omega \xi)u = i\omega Qu + f$$

Show Fredholm property for compactly supported q_ε, q_μ

→ Helmholtz decompositions

Conclusions

The radiation problem for time harmonic Maxwell's equations in wave-guides can be solved

- ▶ Method: Functional analysis (implicit function theorem)
- ▶ Underlying operator is Fredholm (for fixed quasi-moment α)

Further results:

- ▶ Compactly perturbed media
- ▶ $Y = B$

Thank you!

Recent articles:

Kirsch A and Schweizer B (2024), "Time harmonic Maxwell's equations in periodic waveguides" (submitted)

Kirsch A and Schweizer B (2024), "Periodic wave-guides revisited: Radiation conditions, limiting absorption principles, and the space of bound states solutions", *Mathematical Methods in the Applied Sciences*