

Meta-Materials for light: Homogenization of Maxwell equations

Workshop “Variational Views in Mechanics and Materials”

Ben Schweizer

TU Dortmund

Pavia — June 25, 2013

Is $\frac{7}{6}$ a small number?

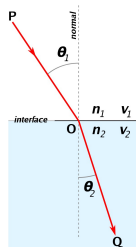
Shortest Paths



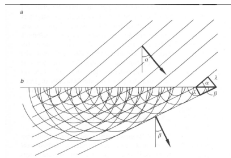
Fermat's principle of
the fastest path:

Light finds the
 fastest way to reach
 the destination,

$$\frac{\sin \Theta_1}{\sin \Theta_2} = \frac{v_1}{v_2} = \frac{n_2}{n_1}$$



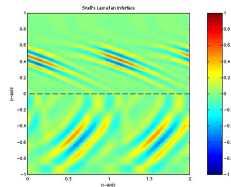
Wave equation



Huygens' principle
 of **superpositions**

Wave equation

$$\partial_t^2 u = \Delta u$$



Numerical solution

Maxwell's Equations

Variables:

- ▶ Electric field E
- ▶ Magnetic field H

Simplification:

- ▶ Time harmonic solutions

$$H, E \sim e^{-i\omega t}$$

Remarks:

- ▶ Vacuum: $\mu = \varepsilon = 1$
- ▶ Material parameter

$$\text{Im } \varepsilon \leftrightarrow \text{conductivity}$$

Maxwell's Equations

$$\text{curl } E = i\omega\mu H$$

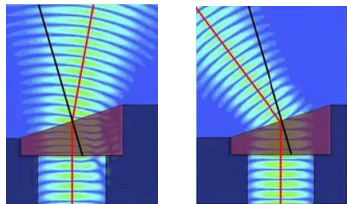
$$\text{curl } H = -i\omega\varepsilon E$$

Negative index of refraction

Veselago (1968)

Properties of materials with negative index, Maxwell equations

If $n_1 > 0$ and $n_2 < 0$, then light should be refracted “backward”.

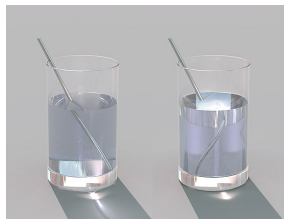


Solutions for positive and negative index

But ... in Maxwell's Equations

- ▶ $\text{Re } \epsilon < 0$ possible
- ▶ μ is always 1
- ▶ $\text{Re } \mu\epsilon < 0$: light can not travel in the medium

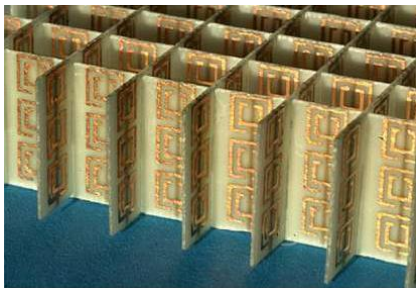
Negative Index: ϵ and μ negative!



Computer graphics: Negative refraction

Experimental construction of Meta-Materials

- ▶ Pendry et al. (~ 2000) suggest a split ring construction



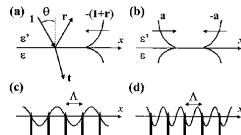
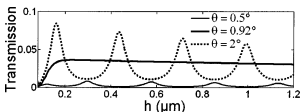
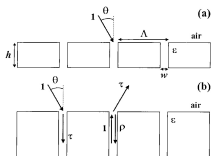
A negative index meta-material

- ▶ Experiments confirm the negative index

Transmission through sub-wavelength holes

Observation: Light hits a metal layer with holes. Even though the holes have sub-wavelength dimensions, the light can exit at the other side.

- ▶ Ebbesen, T. W. and Lezec, H. J. and Ghaemi, H. F. and Thio, T. and Wolff, P. A. *Extraordinary optical transmission through sub-wavelength hole arrays*, Letters to Nature 391, 1998.
- ▶ Porto, J. A. and Garcia-Vidal, F. J. and Pendry, J. B. *Transmission Resonances on Metallic Gratings with Very Narrow Slits*, Phys. Rev. Lett. 14, 1999.
- ▶ Mary, A. and Rodrigo, Sergio G. and Martin-Moreno, L. and Garcia-Vidal, F. J., *Holey metal films: From extraordinary transmission to negative-index behavior*, Physical Review B 80, 2009.
- ▶ Cao, Qing and Lalanne, Philippe *Negative Role of Surface Plasmons in the Transmission of Metallic Gratings with Very Narrow Slits*, Phys. Rev. Lett. 5, 2002.
- ▶ P. Lalanne, C. Sauvan, J. P. Hugonin, J. C. Rodier, and P. Chavel *Perturbative approach for surface plasmon effects on flat interfaces periodically corrugated by subwavelength apertures* Physical Review B 68, 2003



Surface plasmons: A mathematicians view

We consider the Helmholtz equation $\nabla \cdot (a(x)\nabla u(x)) = -k^2 u(x)$ or even

$$\nabla \cdot (a(x)\nabla u(x)) = 0$$

Let a be $+1$ for $x_1 > 0$ and -1 for $x_1 < 0$, $\omega > 0$ arbitrary

$x_1 < 0$	$x_1 > 0$
$a = -1$	$a = +1$
$u(x_1, x_2) = \exp(\omega x_1) \sin(\omega x_2)$	$u(x_1, x_2) = \exp(-\omega x_1) \sin(\omega x_2)$

Then u is	harmonic
and	continuous
and $a(x)\partial_{x_1} u(x)$ is	continuous

Similarly, solutions of the Helmholtz equation can be obtained
 → we found a wave solution that localizes at the interface

Time-harmonic Maxwell Equations

Later, $\eta > 0$ stands for the size of the holes ...

$$\operatorname{curl} E_\eta = i\omega\mu_0 H_\eta$$

$$\operatorname{curl} H_\eta = -i\omega\varepsilon_\eta\varepsilon_0 E_\eta$$

Wave number k and wavelength $\lambda = \frac{2\pi}{k}$. Further assumptions:

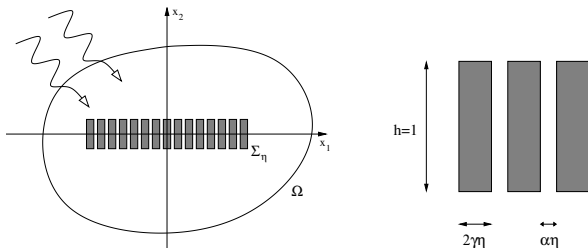
- ▶ invariance in direction x_3
- ▶ magnetic transverse polarization $H = (0, 0, u)$

2D Helmholtz equation for $H_\eta = (0, 0, u_\eta)$ with $u_\eta = u_\eta(x_1, x_2)$

$$\nabla \cdot \left(\frac{1}{\varepsilon_\eta} \nabla u_\eta \right) = -k^2 u_\eta.$$

Question: What is the behavior of u_η in the limit $\eta \rightarrow 0$?

Geometry and permittivity



The domain Ω : Maxwell equations are solved
Rectangle R : The original shape of the metal
Union of small rectangles Σ_η : The metal part after cutting holes

$$\varepsilon_\eta(x) = \begin{cases} \frac{\varepsilon_r}{\eta^2} & \text{for } x \in \Sigma_\eta \\ 1 & \text{for } x \notin \Sigma_\eta \end{cases}$$

Note: $|\varepsilon_\eta|$ is huge in the metal part!

First thoughts on the system

With $a_\eta = 1/\varepsilon_\eta$ (order η^2 in the metal) we must study

$$\nabla \cdot (a_\eta \nabla u_\eta) + k^2 u_\eta = 0.$$

1. Outside the metal: $a_\eta \equiv 1 \rightarrow$ no oscillations $\rightarrow \nabla_y u = 0$
2. In the metal: $a_\eta = \eta^2 \varepsilon_r^{-1} \rightarrow \nabla_y \cdot (\varepsilon_r^{-1} \nabla_y u) + k^2 u = 0$

With aperture $\alpha \in (0, 1)$ and metal thickness $2\gamma = 1 - \alpha$:
 Define $\Psi : \mathbb{R} \rightarrow \mathbb{C}$ as the continuous, 1-periodic solution of

$$\begin{aligned} \partial_z^2 \Psi(z) &= -k^2 \varepsilon_r \Psi(z) && \text{for } z \in (-\gamma, \gamma) && \text{(metal)} \\ \Psi(z) &= 1 && \text{for } z \in [-1/2, 1/2] \setminus (-\gamma, \gamma) && \text{(void)} \end{aligned}$$

Ψ is given by

$$\Psi(z) = \begin{cases} \frac{\cosh(k\sigma z)}{\cosh(k\sigma\gamma)} & \text{for } |z| \leq \gamma \\ 1 & \text{for } \gamma < |z| \leq 1/2 \end{cases}$$

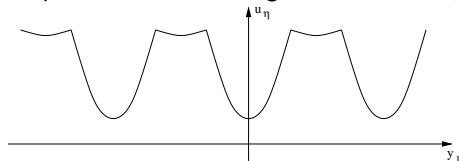
Qualitative behavior of solutions

We expect (in the rectangle R):

$$u_\eta \approx U(x_1, x_2)\Psi(x_1/\eta)$$

In particular, in the single slit:

$$u_\eta \approx U(x_2)$$



Here: $\varepsilon_r < 0$ real

Equation in the single slit is (for some $\tau \in \mathbb{C}$)

$$\frac{\partial^2}{\partial x_2^2} U = -k^2 \tau^2 U$$

Qualitative Argument: The second derivative $\partial_{x_1}^2 u_\eta$ in the slit is proportional to the values at the metal interfaces.

- ▶ Solutions $U(x_2)$ are $\cos(\tau k x_2)$ and $\sin(\tau k x_2)$
- ▶ For height $h > 0$ in resonance with τ : Upper and lower boundary coupled

The effective system

Original system: with $a_\eta = 1/\varepsilon_\eta$

$$\nabla \cdot (a_\eta \nabla u_\eta) + k^2 u_\eta = 0 \quad \text{in } \Omega$$

Limit system: (loosely stated)

$$\nabla \cdot (a_{\text{eff}} \nabla U) + k^2 \mu_{\text{eff}} U = 0 \quad \text{in } \Omega$$

$a_{\text{eff}} : \mathbb{R}^2 \rightarrow \mathbb{R}^{2 \times 2}$ and $\mu_{\text{eff}} : \mathbb{R}^2 \rightarrow \mathbb{C}$ are effective coefficients

$$a_{\text{eff}}(x) := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \mu_{\text{eff}}(x) := 1 \quad \text{for } x \in \mathbb{R}^2 \setminus R$$

$$a_{\text{eff}}(x) := \begin{pmatrix} 0 & 0 \\ 0 & \alpha \end{pmatrix} \quad \text{and} \quad \mu_{\text{eff}}(x) := \beta \quad \text{for } x \in R$$

$\alpha > 0$: aperture volume = relative slit width

$$\beta := \int_{-1/2}^{1/2} \Psi(z) dz = \frac{2 \sinh(k\sigma\gamma)}{k\sigma \cosh(k\sigma\gamma)} + \alpha \in \mathbb{C}.$$

Theorem (Bouchitté and S., 2012)

Assume $\beta \neq 0$, $\varepsilon_r \neq 0$, $a_\eta := \varepsilon_\eta^{-1}$ (small in the metal). Consider

$$\nabla \cdot j_\eta = -k^2 u_\eta, \quad j_\eta = a_\eta \nabla u_\eta,$$

with limits $u_\eta \rightharpoonup u$ and $j_\eta \rightharpoonup j$ in $L^2(\Omega)$ for $\eta \rightarrow 0$. Set

$$U(x) := \begin{cases} u(x) & \text{for } x \in \Omega \setminus R \\ \beta^{-1} u(x) & \text{for } x \in R \end{cases}$$

There holds $\partial_{x_2} U \in L^2_{\text{loc}}(\Omega)$ and

$$j = \begin{cases} (\partial_{x_1} U, \partial_{x_2} U) & \text{in } \Omega \setminus \overline{R} \\ (0, \alpha \partial_{x_2} U) & \text{in } R \end{cases}$$

and

$$\nabla \cdot j = -k^2 u \text{ in } \Omega$$

Method of proof

1.) Two-scale convergence: $u_\eta \rightarrow u_0(x_1, x_2, y_1)$ with $(u_\eta \rightharpoonup u)$

$$u_0(x, y) = \begin{cases} u(x) & \text{for } x \notin R \\ \beta^{-1} u(x) \Psi(y_1) & \text{for } x \in R \end{cases}$$

2.) Two-scale convergence: $j_\eta \rightarrow j_0(x, y)$ with $(j_\eta \rightharpoonup j)$

$$j_0(x, y) = \begin{cases} j(x) & \text{for } x \notin R \\ \alpha^{-1} j_2(x) e_2 \mathbf{1}_{\{|y_1| > \gamma\}} & \text{for } x \in R \end{cases}$$

3.) The distributional derivatives of U satisfies $\partial_{x_2} U \in L^2(\Omega)$.
 Furthermore

$$j(x) = \begin{cases} (\partial_{x_1} U(x), \partial_{x_2} U(x)) & \text{for } x \notin \bar{R} \\ (0, \alpha \partial_{x_2} U(x)) & \text{for } x \in R \end{cases}$$

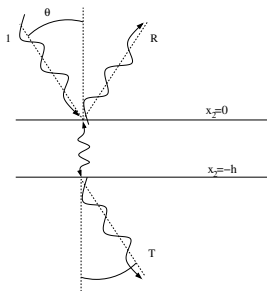
To show this relation: Consider only the void!

Transmission properties

Ansatz ($R \in \mathbb{C}$ for reflection, $T \in \mathbb{C}$ for transmission):

$$U(x_1, x_2) = \begin{cases} e^{ik(\sin(\theta)x_1 - \cos(\theta)x_2)} + Re^{ik(\sin(\theta)x_1 + \cos(\theta)x_2)} & \text{for } x_2 > 0, \\ (A_1 \cos(\tau k x_2) + A_2 \sin(\tau k x_2)) e^{ik(\sin(\theta)x_1} & \text{for } 0 > x_2 > -h \\ Te^{ik(\sin(\theta)x_1 - \cos(\theta)(x_2+h))} & \text{for } -h > x_2. \end{cases}$$

$\tau := \sqrt{\beta/\alpha}$ reflects the equation $\partial_{x_2}^2 U = -k^2 \tau^2 U$ in the structure



At the (horizontal) interfaces:

- ▶ continuity of U
- ▶ continuity of $j \cdot e_2$

Results on transmission properties

A calculation with the **transfer matrix** provides for $T \in \mathbb{C}$

$$T = \left(\cos(\tau kh) - \frac{i}{2} \left[\frac{\alpha\tau}{\cos(\theta)} + \frac{\cos(\theta)}{\alpha\tau} \right] \sin(\tau kh) \right)^{-1}$$

Physical values taken from Qing and Lalanne in non-dimensional form ($h = 1$):

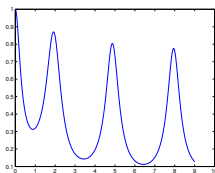
$\eta = 7/6$	$\alpha = 1/7$	$\gamma = 3/7$
$\lambda = 15/6$	$k = 2\pi/\lambda \approx 2.51$	$\varepsilon_\eta = (0.12 + 3.7i)^2$

Explicit formulas for

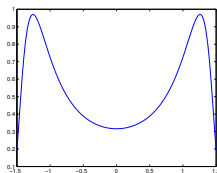
$$\beta = \beta(\sigma, k, \alpha)$$

$$\tau = \tau(\beta, \alpha)$$

$$T = T(\tau, k)$$



Left: $T = T(k)$ for normal incidence, $\theta = 0$.



Right: $T = T(\theta)$ for wave-number $k = 0.8$.

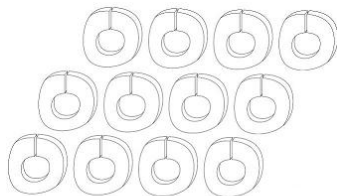
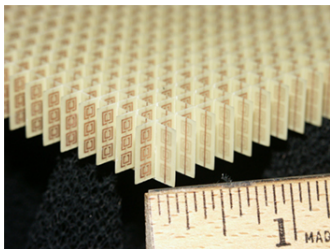
Conclusions on transmission

- ▶ We analyze: metal with large permittivity (η^{-2}) and small holes (η)
- ▶ Effective system is Maxwell-type, permittivity in x_1 -direction is $+\infty$
- ▶ Natural field is U , not u (field outside the metal).
The field U satisfies the continuity condition!
- ▶ Effective equations show the astonishing transmission $T \approx 1$

Negative index Meta-Materials

- ▶ Lamacz, A. and Schweizer, B., SIAM J. Math. Anal. 2013
- ▶ Bouchitté, G. and Schweizer, B., SIAM J. Mult. Mod. 2010

$$(\operatorname{Re} \mu < 0)$$



$$\Sigma_\eta := \bigcup_k \eta(k + \Sigma_Y^\eta)$$

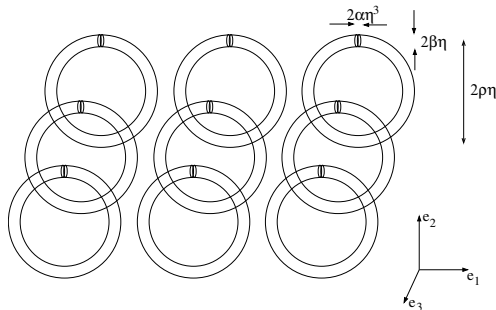
A negative index material in experiments ... and in mathematics

(H_η, E_η) solves the Maxwell system with a radiation condition at ∞ .

$$\begin{aligned}\operatorname{curl} E_\eta &= i\omega H_\eta \\ \operatorname{curl} H_\eta &= -i\omega \varepsilon_\eta E_\eta\end{aligned}$$

Microscopic geometry

“Many rings with thin slits”



The material parameter is

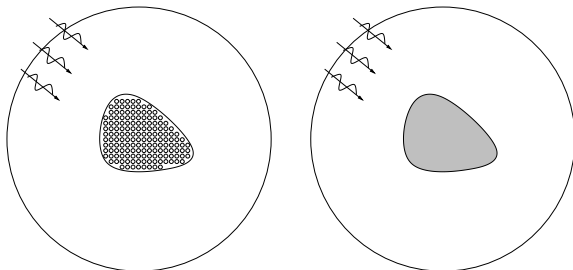
$$\varepsilon_{\eta} = \begin{cases} 1 + i \frac{\kappa}{\eta^2} & \text{in the rings} \\ 1 & \text{else} \end{cases}$$

The parameter η appears 3×:

1. thin rings / many rings
2. high conductivity
3. very thin slit

Homogenization

The aim is to replace the **complex structure** of *many* split rings with *high conductivity* by a **homogeneous Meta-material**.



The resulting equations are of the form

$$\operatorname{curl} E = i\omega\mu_{\text{eff}}H$$

$$\operatorname{curl} H = -i\omega\varepsilon_{\text{eff}}E$$

For appropriate parameters holds $\operatorname{Re}(\mu_{\text{eff}}) < 0$.

Main result

Let ε_η be given with $\varepsilon_\eta = 1 + i\frac{\kappa}{\eta^2}$ in the rings.

Let (H_η, E_η) solve the Maxwell system with a radiation condition at ∞ .

$$\operatorname{curl} E_\eta = i\omega H_\eta$$

$$\operatorname{curl} H_\eta = -i\omega\varepsilon_\eta E_\eta$$

Theorem (Bouchitté – S. 2010, Lamacz – S. 2013)

Let $(H_\eta, E_\eta) \rightharpoonup (H, E)$ in $L^2_{loc}(\mathbb{R}^3)$ for $\eta \rightarrow 0$. Then, for matrices $\hat{\mathcal{M}}_\lambda$ and $\hat{\mathcal{N}}$, the limit functions solve

$$\operatorname{curl} E = i\omega H$$

$$\operatorname{curl} (\hat{\mathcal{M}}_\lambda H) = -i\omega \hat{\mathcal{N}} E.$$

There holds $\hat{\mathcal{M}}_\lambda = \mathcal{M}_0 + \lambda(\omega, \kappa)m_0 e_3 \otimes e_3$ in Ω , with

$$\lambda(\omega, \kappa) = \frac{-\varepsilon_0 \mu_0 \omega^2 D_3(\omega, \kappa)}{\alpha(\pi\rho)^{-1} + \varepsilon_0 \mu_0 \omega^2 D_0(\omega, \kappa) - i\kappa^{-1}}.$$

Interpretation: The physical field is $\hat{H} = \hat{\mathcal{M}}_\lambda H$ such that $H = \mu^{\text{eff}} \hat{H}$

Formally, 1D-rings

Thin rings:

$$Y = (0, 1)^3$$

$$\Sigma = S^1 \subset Y$$

$$\dim(\Sigma) = 1$$

Shape of J :

$$j_\eta := \eta \varepsilon_\eta E_\eta \rightharpoonup J$$

$$\text{supp } J \subset \Sigma$$

$$\text{div } J = 0$$

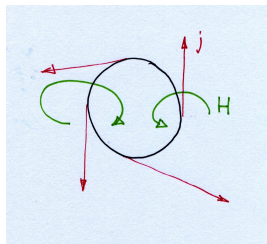
$$\Rightarrow J = j \tau \mathcal{H}^1 \llcorner \Sigma.$$

$$\text{curl } E_\eta = i\omega H_\eta$$

$$\text{curl } H_\eta = -i\omega \varepsilon_\eta E_\eta$$

Shape of H :

$$\text{curl } H = J$$



This determines the non-trivial part of the magnetic field.

Homogenization procedure

$$\begin{aligned}\operatorname{curl} E_\eta &= i\omega H_\eta \\ \operatorname{curl} H_\eta &= -i\omega \varepsilon_\eta E_\eta\end{aligned}$$

Two-scale convergence: $H_\eta(x) \rightarrow H_0(x, y)$ and $E_\eta(x) \rightarrow E_0(x, y)$ in the sense of two-scale convergence (L^2 or measures)

Loose definition: In the single periodicity cell $Y = [0, 1]^3$ the solution looks like

$$H_\eta(x) \sim H_0(x, y), \quad E_\eta(x) \sim E_0(x, y),$$

where $y \in Y$ is the local position within the cell.

The limits $H_0(x, \cdot)$ and $E_0(x, \cdot)$ solve the Maxwell equations, for example

$$\begin{aligned}\operatorname{div}_y H_0(x, y) &= 0 \text{ in } Y, & \operatorname{curl}_y E_0(x, y) &= 0 \text{ in } Y, \\ \operatorname{curl}_y H_0(x, y) &= 0 \text{ in } Y \setminus \Sigma.\end{aligned}$$

The current J

$$\begin{aligned}\operatorname{curl} E_\eta &= i\omega H_\eta \\ \operatorname{curl} H_\eta &= -i\omega \varepsilon_\eta E_\eta\end{aligned}$$

We additionally consider the field

$$J_\eta := \eta \varepsilon_\eta E_\eta \rightarrow J_0(x, y).$$

Then

$$\begin{aligned}\operatorname{div} J_\eta = 0 &\text{ implies } \operatorname{div}_y J_0(x, \cdot) = 0 \text{ in } Y \\ E_\eta \text{ bounded} &\text{ implies } J_0(x, \cdot) = 0 \text{ in } Y \setminus \Sigma \\ \eta \operatorname{curl} H_\eta = -i\omega J_\eta &\text{ implies } \operatorname{curl}_y H_0(x, \cdot) = -i\omega J_0(x, \cdot) \text{ in } Y.\end{aligned}$$

Outline of the homogenization proof

1. Introduce **current** $J_\eta := \eta \varepsilon_\eta E_\eta$ and derive estimates
2. Consider two-scale limits H_0, E_0, J_0 and **derive cell problems**
Difficulty: Slit vanishes
3. **Analyze cell problems.**
Difficulty: construction of the special solution H^0 “pointing through the ring”
4. Write the two-scale limit as

$$H_0(x, y) = j(x)H^0(y) + \sum_{k=1}^3 H_k(x)H^k(y),$$

and **determine** j from the slit

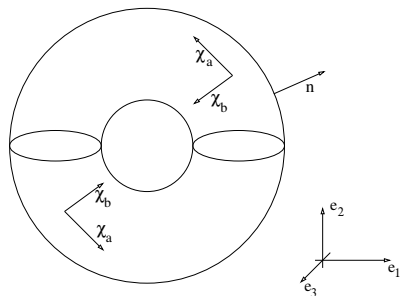
5. Conclude the **macroscopic equation**

Result of 4: $j(x) = \lambda H_3(x)$ with

$$\lambda(\omega, \kappa) = \frac{-\varepsilon_0 \mu_0 \omega^2 D_3(\omega, \kappa)}{\alpha(\pi \rho)^{-1} + \varepsilon_0 \mu_0 \omega^2 D_0(\omega, \kappa) - i\kappa^{-1}}.$$

Step 3: 3D-cell problem without slit

We must study the 3D cell problem:



Σ the torus,
 $Y = (0, 1)^3$,

The H -problem

$$\begin{aligned}\operatorname{curl}_y H + i\omega\varepsilon_0 J &= 0 \text{ in } Y, \\ \operatorname{div}_y H &= 0 \text{ in } Y, \\ H &\text{ is periodic in } Y,\end{aligned}$$

is coupled to the J -problem

$$\begin{aligned}\operatorname{curl}_y J + \kappa\omega\mu_0 H &= 0 \text{ in } \Sigma, \\ \operatorname{div}_y J &= 0 \text{ in } Y, \\ J &= 0 \text{ in } Y \setminus \bar{\Sigma}.\end{aligned}$$

Lemma (Bouchitté – S., 2010)

The solution space to the above problem is four-dimensional.

Idea for the Lemma

Space for solutions:

$$X_0 := \{u \in L^2_{per}(Y) : \operatorname{div} u \in L^2(Y), \operatorname{curl} u = 0 \text{ on } Y \setminus \Sigma\}$$

Bilinear form:

$$b_0(u, v) := \int_Y \operatorname{div} u \operatorname{div} \bar{v} - ik_0^2 \int_Y u \bar{v}$$

Regarding normalization:

On the 3D-torus $\Sigma \subset \mathbb{R}^3$ exists a vector field χ_a such that with

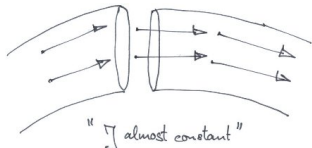
$$\operatorname{curl} \chi_a = 0, \quad \exists \Phi : \chi_a = \nabla \Phi.$$

Normalize special solution u with

$$\int_Y u \cdot e_k = 0 \text{ for } k = 1, 2, 3, \text{ and } \int_{\Sigma} u \cdot \chi_a = 1.$$

Step 4: Slit analysis

The flux J_η is almost constant — across the slit!
(despite $\varepsilon_\eta = 1 + i\frac{\kappa}{\eta^2}$ in the ring)



$$E_\eta \sim \begin{cases} \frac{1}{i\omega} \frac{1}{\eta} j_0 \tau & \text{in the slit} \\ \frac{-1}{\kappa\omega} \eta j_0 \tau & \text{inside the material} \end{cases}$$

$$\int_{\text{closed ring}} E_\eta \cdot \tau = \int_{\text{disc}} e_3 \cdot \text{curl } E_\eta = i\omega \int_{\text{disc}} e_3 \cdot H_\eta$$

... and in the limit $\eta \rightarrow 0$:

$$-\frac{2i}{\omega} \left(\alpha - i\frac{\rho\pi}{\kappa} \right) j_0(x) = i\omega (D_3 H_3(x) - D_0 j_0(x))$$

This provides $j_0(x) = \lambda(\omega) H_3(x)$.

Step 5: Macroscopic equation

Use test-functions Φ from $\Phi(x, y) = \psi(x)\Theta(y)$ with $\text{curl}_y \Theta = 0$ in Y , $\Theta \equiv 0$ on $\text{conv } \Sigma$. Then, for $\eta \rightarrow 0$,

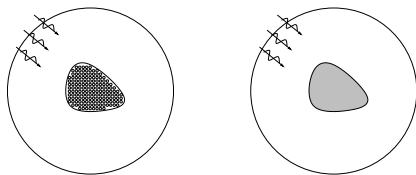
$$\int_{\mathbb{R}^3} \text{curl } H_\eta \cdot \Phi = \int_{\mathbb{R}^3} (-i\omega \varepsilon_\eta E_\eta) \cdot \Phi \rightarrow \langle -i\omega E(x), \psi(x) \rangle$$

... and the left hand side equals

$$\begin{aligned} \int_{\mathbb{R}^3} H_\eta \cdot \text{curl } \Phi &= \int_{\mathbb{R}^3} H_\eta(x) \Theta(x/\eta) \wedge \nabla \psi(x) dx \\ &\rightarrow \int_{\mathbb{R}^3} \int_Y \{H(x) + \lambda(\omega) H_3(x) H^0(y)\} \Theta(y) \wedge \nabla \psi(x) dy dx \\ &= \int_{\mathbb{R}^3} [\hat{M}H(x)] \wedge \nabla \psi(x) dx = \langle \text{curl } [\hat{M}H], \psi(x) \rangle \end{aligned}$$

This provides $\text{curl } [\hat{M}H] = -i\omega E(x)$.

Conclusions on negative index materials



$$\begin{aligned}\text{curl } E &= i\omega\mu_{\text{eff}}H \\ \text{curl } H &= -i\omega\varepsilon_{\text{eff}}E\end{aligned}$$

- ▶ split ring geometry with highly conducting rings
- ▶ 3D-scattering problem, Maxwell equations
- ▶ formulas for μ_{eff} and ε_{eff} in terms of conductivity and geometric quantities
- ▶ mathematical proofs for the homogenization result
- ▶ Calculations show: μ_{eff} can be negative (despite $\mu_0 \equiv 1$)

Thank you!

All formulas (perfect transmission):

Permittivity with $\sigma^2 = -\varepsilon_r$

$$\varepsilon_\eta(x) = \begin{cases} \frac{\varepsilon_r}{\eta^2} & \text{for } x \in \Sigma_\eta \\ 1 & \text{for } x \notin \Sigma_\eta \end{cases}$$

Use $a_\eta = 1/\varepsilon_\eta$

$$\beta = \frac{2 \sinh(k\sigma\gamma)}{k\sigma \cosh(k\sigma\gamma)} + \alpha$$

$$\tau = \sqrt{\beta/\alpha}$$

$$T = \left(\cos(\tau kh) - \frac{i}{2} \left[\frac{\alpha\tau}{\cos(\theta)} + \frac{\cos(\theta)}{\alpha\tau} \right] \sin(\tau kh) \right)^{-1}$$