Homogenization of wave equations on long time scales and dispersive effective equations

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Physical origins of the wave equation

Acoustics

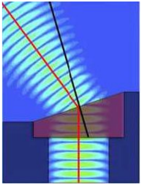
Pressure in an ideal gas

$$\rho_0 \partial_t v + p_0'(\rho_0)\,\nabla \rho = 0$$
$$\partial_t \rho + \rho_0 \nabla \cdot v = 0$$

Optics

Maxwell's equations

$$\partial_t(\mu H) = -\mathrm{curl}\ E$$
$$\partial_t(\varepsilon E) = \mathrm{curl}\ H$$



Elastic media

Equations of elasticity

$$\rho \partial_t^2 u + \nabla \cdot \sigma = 0$$
$$\sigma = A \nabla^s u$$

In simplified settings, each model leads to the wave equation:

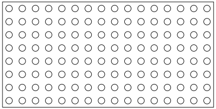
Wave equation

$$\rho \partial_t^2 u = \nabla \cdot (a \nabla u)$$

with coefficients

$\rho = \rho(x)$ and $a = a(x)$

Homogenization of the wave equation



Let $a : \mathbb{R}^d \to (\delta, \infty)$ be 1-periodic, $a_\varepsilon(x) := a(x/\varepsilon)$

Homogenization problem

$$\partial_t^2 u^\varepsilon(x) = \nabla \cdot (a_\varepsilon(x) \nabla u^\varepsilon(x))$$

Classical homogenization: $u^\varepsilon \approx u$, where u solves

Homogenized equation

$$\partial_t^2 u(x,t) = \nabla \cdot a_* \nabla u(x,t)$$

Dimension 1: Original problem: $\partial_t^2 u^\varepsilon = \partial_x (a(x/\varepsilon) \partial_x u^\varepsilon)$

Homogenized equation: $\partial_t^2 u(x,t) = c^2 \partial_x^2 u(x,t)$

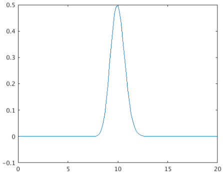
With initial conditions $u(x,0) = f(x)$ and $\partial_t u(x,0) = 0$, the exact solution is given by

$$u(x,t) = \tfrac{1}{2} f(x - ct) + \tfrac{1}{2} f(x + ct)$$

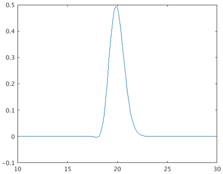
Observation of dispersion in heterogeneous media

Homogenized equation $\partial_t^2 u = \partial_x^2 u$ with solution $u(x,t) = f(x-t)$

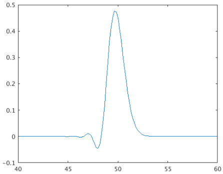
Numerical results in periodic medium, $\varepsilon = 1/6$



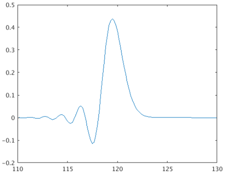
Time $t = 10$. Pulse centered at $x = 10$



At time $t = 20$, essentially unchanged



At time $t = 50$



At time $t = 120$

J. Fish, W. Chen, and G. Nagai. Nonlocal dispersive model for wave propagation in heterogeneous media, 2002.

Dispersion — elementary analysis

Planar-wave ansatz: $u(x,t) = e^{ik\cdot x}e^{-i\omega t}$

$$u \text{ solves } \partial_t^2 u = c^2 \Delta u \iff \omega^2 = c^2|k|^2$$

Dispersion relation: $\omega(k) = \pm c|k|$

Re-combine to find solutions: (here: $x \in \mathbb{R}$)

$$u(x,t) = \int_{\mathbb{R}} \hat{f}_\pm(k) e^{ikx \mp ic|k|t}\, dk \text{ is the general solution}$$

- Satisfy initial conditions by choosing $\hat{f}_\pm(k)$
- The dispersion relation yields a representation formula

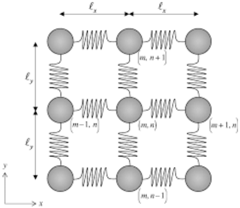
No dispersion: $\omega(k) = \pm c|k|$

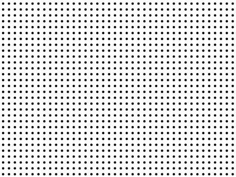
Solutions *depend only on* $x + ct$ and on $x - ct$ **($\rightarrow$ only shifts)**

Opposite case (c depends on k): dispersive behavior

Plane waves travel with different speeds.

Lattice dynamics





Periodicity of the lattice: $\varepsilon > 0$
Dimension $d \geq 1$
Lattice points $\gamma \in (\varepsilon\mathbb{Z})^d$
Displacement $u^\varepsilon(\gamma, t)$

Wave equation

$$\partial_t^2 u^\varepsilon(\gamma, t) = \frac{1}{\varepsilon^2} \sum_{j \in \mathbb{Z}^d} a_j u^\varepsilon(\gamma + \varepsilon j, t)$$

Initial conditions: $u^\varepsilon(\gamma, 0) = u_0(\gamma)$
and $\partial_t u^\varepsilon(\gamma, 0) = u_1(\gamma)$

One-dimensional example: $a_1 = a_{-1} = 1$, $a_0 = -2$, and $a_j = 0$ else

$$\partial_t^2 u^\varepsilon(\gamma) = \frac{1}{\varepsilon^2} \left(u^\varepsilon(\gamma + \varepsilon) - 2u^\varepsilon(\gamma) + u^\varepsilon(\gamma - \varepsilon)\right)$$

Explicit solutions in Fourier space

Fourier transform: $\hat{u}^\varepsilon(k,t) := \varepsilon^d \sum_{\gamma\in(\varepsilon\mathbb{Z})^d} e^{-ik\cdot\gamma} u^\varepsilon(\gamma,t)$ for $k \in \mathbb{R}^d$.

$$\partial_t^2 \hat{u}^\varepsilon(k,t) = \varepsilon^d \sum_{\gamma\in(\varepsilon\mathbb{Z})^d} e^{-ik\cdot\gamma} \frac{1}{\varepsilon^2} \sum_{j\in\mathbb{Z}^d} a_j u^\varepsilon(\gamma+\varepsilon j)$$

$$= \varepsilon^d \frac{1}{\varepsilon^2} \sum_{j\in\mathbb{Z}^d} a_j e^{i\varepsilon k\cdot j} \sum_{\gamma\in(\varepsilon\mathbb{Z})^d} e^{-ik\cdot(\gamma+\varepsilon j)} u^\varepsilon(\gamma+\varepsilon j) = -\frac{\omega(\varepsilon k)^2}{\varepsilon^2} \hat{u}^\varepsilon(k,t)$$

with the dispersion relation

$$\omega(\tilde{k})^2 := -\sum_{j\in\mathbb{Z}^d} a_j e^{i\tilde{k}\cdot j} = -(e^{i\tilde{k}} - 2 + e^{-i\tilde{k}}) = \tilde{k}^2 - \frac{1}{12}\tilde{k}^4 \pm \dots$$

Explicit solution in Fourier space

$$\hat{u}^\varepsilon(k,t) \sim e^{\pm i[\omega(\varepsilon k)/\varepsilon]t}$$

In the example: $\frac{\omega(\varepsilon k)}{\varepsilon} = \sqrt{k^2 - \frac{1}{12}\varepsilon^2 k^4 \pm \dots}$

From dispersion relations to PDEs

We know: All solutions are of the form

$$u(x,t) = \sum_{\pm} \int_{\mathbb{R}^d} \hat{u}_0^{\pm}(k)\, e^{ik\cdot x \mp i\omega(k)t}\; dk$$

with the dispersion relation $\omega^2(k) = k^2 - \frac{\varepsilon^2}{12} k^4$.

We ask: What is the corresponding PDE? Is it

$$\partial_t^2 u = \partial_x^2 u + \frac{\varepsilon^2}{12} \partial_x^4 u \qquad ?$$

Name: "Bad Boussinesq equation"

- Formally correct ...
- ... but not useful: ∂_x^2 is a negative operator, $\frac{\varepsilon^2}{12}\partial_x^4$ positive

Replacement trick: Highest order is $\partial_t^2 u = \partial_x^2 u$, the good equation is

$$\partial_t^2 u = \partial_x^2 u + \frac{\varepsilon^2}{12} \partial_x^2 \partial_t^2 u$$

A.Lamacz. Dispersive effective models for waves in heterogeneous media, 2011

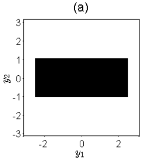
Higher space dimension

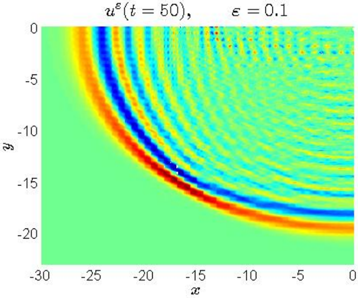
Homogenized system

$$\partial_t^2 w^\varepsilon = AD^2 w^\varepsilon + \varepsilon^2 ED^2 \partial_t^2 w^\varepsilon - \varepsilon^2 FD^4 w^\varepsilon$$

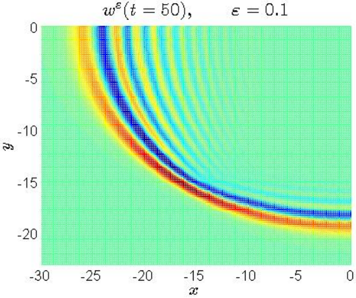
Numerical comparison:

- the original problem with ε-scale
- the weakly dispersive equation





u^ε: solution for coefficient a_ε



w^ε: solution of weakly dispersive equation

Method of proof: Bloch-wave analysis

Proposition (Bloch-wave approximation of u^ε)

$$v^\varepsilon(x,t) := (2\pi)^{-n/2}\,\frac{1}{2}\sum_{\pm}\int_K F_0(k)e^{\mathrm{i}k\cdot x}\exp\left(\pm\mathrm{i}t\sqrt{\sum A_{lm}k_l k_m}\right)$$

$$\times\exp\left(\pm\frac{\mathrm{i}\varepsilon^2}{2}t\frac{\sum C_{lmnq}k_l k_m k_n k_q}{\sqrt{\sum A_{lm}k_l k_m}}\right)\,dk$$

satisfies

$$\sup_{t\in[0,T\varepsilon^{-2}]}\|u^\varepsilon(\cdot,t)-v^\varepsilon(\cdot,t)\|_{(L^2+L^\infty)(\mathbb{R}^n)}\le C_0\varepsilon$$

- F_0 is the Fourier transform of initial data
- Neglect all terms $m\neq 0$ in Bloch expansion
- Taylor expansion of the smallest Bloch eigenvalue $\mu_0(k)$

Proposition and replacement trick provide equation for w^ε

T.Dohnal, A.Lamacz, B.Schweizer. Bloch-wave homogenization on large time scales and dispersive effective wave equations, 2014.

A. Abdulle and T. Pouchon. Effective models for the multidimensional wave equation in heterogeneous media over long time and numerical homogenization, 2016.

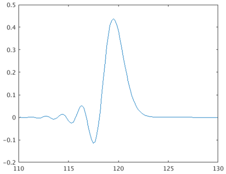
A. Benoit and A. Gloria. Long-time homogenization and asymptotic ballistic transport of classical waves, 2017.

Profile described by a linearized KdV equation

Task: Study the long time behavior of

$$\partial_t^2 u = c^2 \partial_x^2 u + \varepsilon^2 b \, \partial_x^4 u$$

We want an equation for *this* $\longrightarrow$



Solution at time $t = 120$

Ansatz $u(x,t) = V(x - ct, \varepsilon^2 t)$. Inserting yields

$$c^2 \partial_z^2 V - 2c \, \varepsilon^2 \partial_z \partial_\tau V + O(\varepsilon^4) = c^2 \partial_z^2 V + \varepsilon^2 b \, \partial_z \partial_z^3 V$$

Result: Linearized KdV-equation

$$\partial_\tau V(z,\tau) = -\frac{b}{2c} \, \partial_z^3 V(z,\tau)$$

1-dimensional profile approximation

Evolution given by a linearized KdV equation; direction $q = \pm 1$:

$$\partial_\tau V^\varepsilon(z,\tau;q) = b(q)\,\partial_z^3 V^\varepsilon(z,\tau;q)$$

Initial condition:

$$\hat{V}_0^\varepsilon(\xi;+1) := \begin{cases} \hat{u}_0^\varepsilon(\xi) & \text{for } \xi > 0 \\ 0 & \text{else} \end{cases}$$

$$\hat{V}_0^\varepsilon(\xi;-1) := \begin{cases} \hat{u}_0^\varepsilon(-\xi) & \text{for } \xi > 0 \\ 0 & \text{else} \end{cases}$$

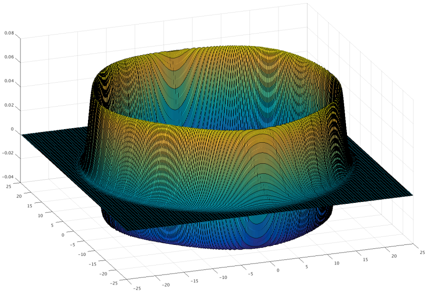
Reconstruction:

$$v^\varepsilon(x,t) := \begin{cases} V^\varepsilon\left(|x| - ct, \varepsilon^2 t; +1\right) & \text{for } x > 0 \\ V^\varepsilon\left(|x| - ct, \varepsilon^2 t; -1\right) & \text{for } x < 0 \end{cases}$$

Theorem (Lattice solution and profile solution)

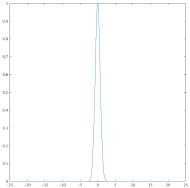
$$\left\| \hat{u}^\varepsilon(.,\tau/\varepsilon^2) - \hat{v}^\varepsilon(.,\tau/\varepsilon^2) \right\|_{L^2(\mathbb{R})} \to 0 \quad \textit{as } \varepsilon \to 0$$

2-dimensional solution

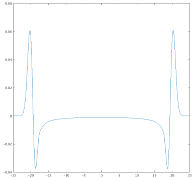


The 2-dimensional solution $x \mapsto u(x, t_0)$ of the lattice equations

$\varepsilon = 1/6$, $x \in (-25, 25)^2$, $t_0 = 20$,
$u_0(x) = e^{-|x|^2}$



Initial values u_0



Solution at time $t_0 = 20$
time step $5 \cdot 10^{-5}$

2-dimensional profile equation

The evolution is given by a linearized KdV equation:

$$\partial_\tau V^\varepsilon(z,\tau;q) = b(q)\,\partial_z^3 V^\varepsilon(z,\tau;q)$$

Initial condition in $d=2$:

$$\hat{V}_0^\varepsilon(\xi;q) := \begin{cases} \sqrt{\frac{\xi}{2\pi i}}\ \hat{u}_0^\varepsilon(\xi q) & \text{for } \xi > 0 \\ 0 & \text{else} \end{cases}$$

Reconstruction:

$$v^\varepsilon(x,t) := \frac{1}{|x|^{(d-1)/2}} V^\varepsilon\left(|x| - ct, \varepsilon^2 t; \frac{x}{|x|}\right)$$

Theorem (B.S. & F.Theil, 2018)

Dimension $d=2$, u^ε the lattice solution, v^ε constructed from KdV-profile. Then $u^\varepsilon \approx v^\varepsilon$.

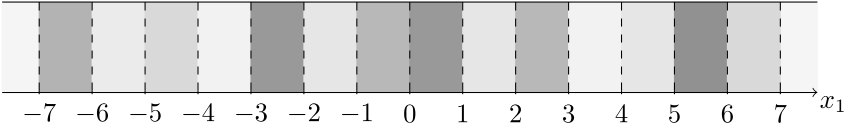
Generalized by A.Lamacz & B.S., 2019: The solution operator

$$\hat{u}_0(k) \mapsto e^{ic|k|\tau/\varepsilon^2} e^{-ib(|k|)\tau} \hat{u}_0(k)$$

acts like $\quad \mathcal{F}_d \circ \mathcal{S} \circ \mathcal{F}_1^{-1} \circ J_b \circ \mathcal{R}$

Stochastic medium. Example: i.i.d. medium

Let $(a_j)_{j\in\mathbb{Z}}$ and $(\rho_j)_{j\in\mathbb{Z}}$ be i.i.d. random variables, e.g.: uniform in $[1,2]$



Set $a(x) = a_j$ and $\rho(x) = \rho_j$ for $x \in [j, j+1)$. Then $a_\varepsilon(x) := a(x/\varepsilon)$

The model has $\bar{\rho} = \langle \rho_0 \rangle$ and $\bar{a} := \langle a_0^{-1} \rangle^{-1}$ and is of class $\gamma = 1/2$

Task: Compare solutions on time intervals $[0, \tau\varepsilon^{-\beta}]$

$u^\varepsilon : \mathbb{R} \times [0,\infty) \times \Omega_{\mathcal{P}} \to \mathbb{R}$ of

$$\rho_\varepsilon \partial_t^2 u^\varepsilon - \partial_x \left(a_\varepsilon \partial_x u^\varepsilon \right) = f$$

$\bar{u} : \mathbb{R} \times [0,\infty) \to \mathbb{R}$ of

$$\bar{\rho} \partial_t^2 \bar{u} - \partial_x \left(\bar{a} \partial_x \bar{u} \right) = f$$

Both with trivial initial conditions, e.g.: $u^\varepsilon(\cdot,0) = \partial_t u^\varepsilon(\cdot,0) = 0$

Time horizon

Always: $f \in C^2(\mathbb{R} \times \mathbb{R}_+, \mathbb{R})$ with compact support and $T_0 > 0$ arbitrary

Definition (Homogenization time parameter β)

Homogenization works with $\beta \geq 0$ if the solutions u^ε and $\bar{u}$ satisfy

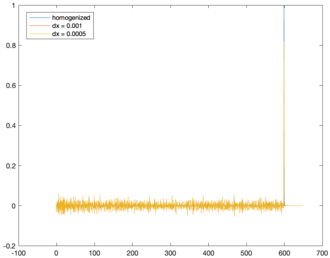
$$\lim_{\varepsilon \to 0} \sup_{t \in [0, T_0 \varepsilon^{-\beta}]} \left\langle \|\partial_t u^\varepsilon(\cdot, t) - \partial_t \bar{u}(\cdot, t)\|_{L^2(\mathbb{R})} \right\rangle = 0$$

Theorem (Main result: Two critical parameters)

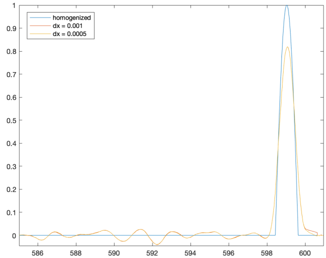
1. *For $\beta < \beta_- := \frac{1-\gamma}{1+\gamma}$: For all coefficients (ρ, a) of class γ,* **homogenization works** *with β*
2. *For $\beta > \beta_+ := \frac{1-\gamma}{\gamma}$: There exist coefficients (ρ, a) of class γ such that* **homogenization does not work** *with β*

Open problem: In the i.i.d. model, homogenization works for $\beta < \frac{1}{3}$ and does not work for $\beta > 1$. What is the critical parameter?

Numerical tests



Solutions on $x \in (0, 700)$



Zoom

Numerical results for $\varepsilon = 0.01$ and $t_0 = 600$ and $x_0 = 600$

Calculations with discretization $dx = 0.001$ and with $dx = 0.0005$

Complexity: 10^6 unknowns, time resolution adds a second dimension that requires the same number of points

Results (formulated for $d = 1$)

- **Long time homogenization:** The weakly dispersive equation $\partial_t^2 w^\varepsilon = \partial_x^2 w^\varepsilon + \varepsilon^2 \partial_t^2 \partial_x^2 w^\varepsilon$ approximates solutions for $t \in (0, T\varepsilon^{-2})$
- **Profile equation for lattices:** The equation $\partial_\tau V = \partial_z^3 V$ describes the evolution of profiles
- **Abstract reconstruction:** Evolutions that are given by factors in Fourier space can be approximated by profiles
- **Stochastic homogenization:** Large time homogenization in a one-dimensional stochastic model

Thank you!

[1] A.Lamacz: Dispersive effective models for waves in heterogeneous media. *Math. Models Methods Appl. Sci. 21(9)*, 2011

[2] T.Dohnal, A.Lamacz, B.S.: Bloch-wave homogenization on large time scales and dispersive effective wave equations. *Multiscale Model. Simul. 12(2)*, 2014

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[4] B.S., F.Theil: Lattice dynamics on large time scales and dispersive effective equations, *SIAM J. Appl. Math. 78(6)*, 2018

[5] A.Lamacz, B.S.: Representation of solutions to wave equations with profile functions, *Analysis and Applications*, 2020

[6] M.Schäffner and B.S., The time horizon for stochastic homogenization of the one-dimensional wave equation, *Asymptotic Analysis*, 2024