

# Homogenization of wave equations and dispersive profiles for ring solutions

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# Mathematical task

- Coefficient  $A$ , strongly elliptic, 1-periodic in every direction
- Seek  $u^\varepsilon : \mathbb{R}^d \times [0, \infty) \rightarrow \mathbb{R}^p$

## Wave system

$$\partial_t^2 u^\varepsilon(x, t) = \nabla \cdot (A(x/\varepsilon) \nabla u^\varepsilon(x, t))$$

- Task: Describe the limit  $\varepsilon \rightarrow 0$       “Homogenization”
- Classical result for  $t \in [0, T_0]$ : There holds  $u^\varepsilon \rightarrow u$  and  $u$  solves the wave equation with the homogenized elliptic operator

- Dispersive effects on time intervals  $t \in [0, T_0 \varepsilon^{-2}]$  ?
- Why rings?
- Stochastic media?

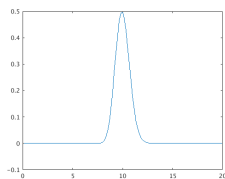
# Observation of dispersion in heterogeneous media

$$\partial_t^2 u^\varepsilon(x, t) = \partial_x (a(x/\varepsilon) \partial_x u^\varepsilon(x, t)) \quad \text{for } x \in \mathbb{R}$$

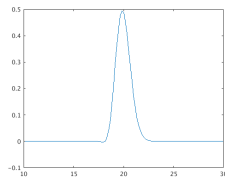
Homogenized equation  $\partial_t^2 u = \partial_x^2 u$  with solution  $u(x, t) = f(x - t)$

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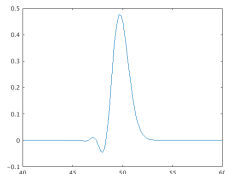
**Numerical results in periodic medium,  $\varepsilon = 1/6$**



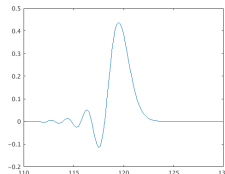
Time  $t = 10$ . Pulse centered at  $x = 10$



At time  $t = 20$ , essentially unchanged

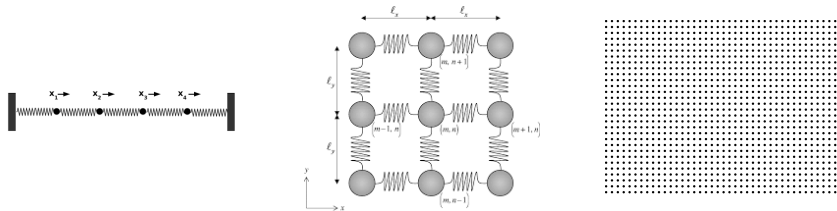


At time  $t = 50$



At time  $t = 120$

# Lattice dynamics



Periodicity of the lattice:  $\varepsilon > 0$

Dimension  $d \geq 1$

Lattice points  $\gamma \in (\varepsilon\mathbb{Z})^d$

Displacement  $u^\varepsilon(\gamma, t)$

Wave equation

$$\partial_t^2 u^\varepsilon(\gamma, t) = \frac{1}{\varepsilon^2} \sum_{j \in \mathbb{Z}^d} a_j u^\varepsilon(\gamma + \varepsilon j, t)$$

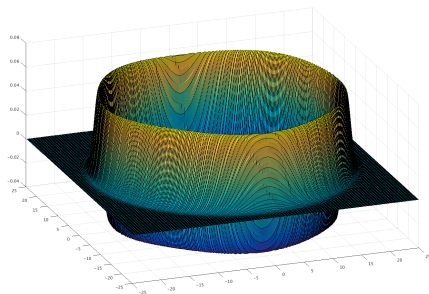
Initial conditions:  $u^\varepsilon(\gamma, 0) = u_0(\gamma)$

and  $\partial_t u^\varepsilon(\gamma, 0) = u_1(\gamma)$

**One-dimensional example:**  $a_1 = a_{-1} = 1$ ,  $a_0 = -2$ , and  $a_j = 0$  else

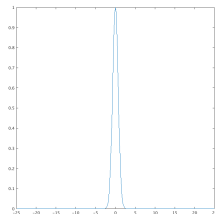
$$\partial_t^2 u^\varepsilon(\gamma) = \frac{1}{\varepsilon^2} (u^\varepsilon(\gamma + \varepsilon) - 2u^\varepsilon(\gamma) + u^\varepsilon(\gamma - \varepsilon))$$

# 2-dimensional solution

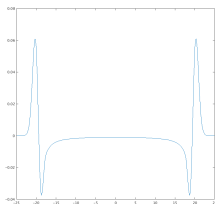


The 2-dimensional solution  $x \mapsto u(x, t_0)$  of the lattice equations

$$\varepsilon = 1/6, \quad x \in (-25, 25)^2, \quad u_0(x) = e^{-|x|^2}$$



Initial values  $u_0$



Solution at time  $t_0 = 20$

# Explicit solutions in Fourier space

Fourier transform:  $\hat{u}^\varepsilon(k, t) := \varepsilon^d \sum_{\gamma \in (\varepsilon\mathbb{Z})^d} e^{-ik \cdot \gamma} u^\varepsilon(\gamma, t)$  for  $k \in \mathbb{R}^d$

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$$\begin{aligned} \partial_t^2 \hat{u}^\varepsilon(k, t) &= \varepsilon^d \sum_{\gamma \in (\varepsilon\mathbb{Z})^d} e^{-ik \cdot \gamma} \frac{1}{\varepsilon^2} \sum_{j \in \mathbb{Z}^d} a_j u^\varepsilon(\gamma + \varepsilon j) \\ &= \varepsilon^d \frac{1}{\varepsilon^2} \sum_{j \in \mathbb{Z}^d} a_j e^{i\varepsilon k \cdot j} \sum_{\gamma \in (\varepsilon\mathbb{Z})^d} e^{-ik \cdot (\gamma + \varepsilon j)} u^\varepsilon(\gamma + \varepsilon j) = -\frac{\tilde{\omega}(\varepsilon k)^2}{\varepsilon^2} \hat{u}^\varepsilon(k, t) \end{aligned}$$

with the **dispersion relation**

$$\tilde{\omega}(\tilde{k})^2 := - \sum_{j \in \mathbb{Z}^d} a_j e^{i\tilde{k} \cdot j}$$

Explicit solution in Fourier space,  $\omega_\varepsilon(k) := \tilde{\omega}(\varepsilon k) / \varepsilon$

$$\hat{u}^\varepsilon(k, t) \sim e^{\pm i\omega_\varepsilon(k)t}$$

Example:  $\omega_\varepsilon(k) = \sqrt{k^2 - \frac{1}{12}\varepsilon^2 k^4 \pm \dots} = |k| - \frac{1}{24}\varepsilon^2 |k|^3 \pm \dots$

# What is dispersion?

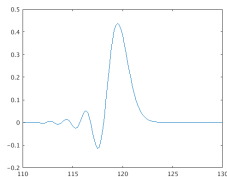
Notation:  $e^{ik \cdot x}$  and  $e^{i\omega t}$ . Example:  $\omega_\varepsilon(k) = |k| - \frac{1}{24}\varepsilon^2|k|^3 \pm \dots$

$c^2\Delta$  has the symbol  $-c^2|k|^2$

**Dispersion** is when  $e^{ic|k|t}$  is replaced by

$$e^{ic|k|t} \cdot e^{-ib|k|^3\varepsilon^2t}$$

**Dispersion** is profile deformations



$$\partial_\tau V(z, \tau) = -b\partial_z^3 V(z, \tau)$$

**Dispersion** is

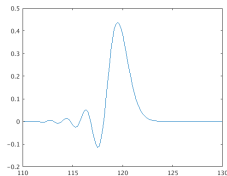
$$\partial_t^2 u = c^2\Delta u + \varepsilon^2(2cb) D^4 u$$

# Profile described by a linearized KdV equation

**Task:** Study the long time behavior of

$$\partial_t^2 u = c^2 \partial_x^2 u + \varepsilon^2 (2cb) \partial_x^4 u$$

We want an equation for *this*  $\rightarrow$



Solution at time  $t = 120$

Ansatz  $u(x, t) = V(x - ct, \varepsilon^2 t)$ . Inserting yields

$$c^2 \partial_z^2 V - 2c \varepsilon^2 \partial_z \partial_\tau V + O(\varepsilon^4) = c^2 \partial_z^2 V + \varepsilon^2 (2cb) \partial_z \partial_z^3 V$$

**Result:** Linearized KdV-equation

$$\partial_\tau V(z, \tau) = -b \partial_z^3 V(z, \tau)$$

# Profile equation for 2D-lattice wave equation

Profile initial condition ( $d = 2$ ) in direction  $q \in S^1$ :

$$\hat{V}_0^\varepsilon(\xi; q) := \begin{cases} \sqrt{\frac{\xi}{2\pi i}} \hat{u}_0^\varepsilon(\xi q) & \text{for } \xi > 0 \\ 0 & \text{else} \end{cases}$$

**Evolution of profile:** Linearized KdV equation:

$$\partial_\tau V^\varepsilon(z, \tau; q) = b(q) \partial_z^3 V^\varepsilon(z, \tau; q)$$

**Reconstruction as ring:**

$$v^\varepsilon(x, t) := \frac{1}{|x|^{(d-1)/2}} V^\varepsilon\left(|x| - ct, \varepsilon^2 t; \frac{x}{|x|}\right)$$

**Theorem (A.Lamacz & B.S. & F.Theil, 2018, 2020)**

*$u^\varepsilon$  the lattice solution,  $v^\varepsilon$  constructed from KdV-profile. Then  $u^\varepsilon \approx v^\varepsilon$ .*

B.S., F.Theil: Lattice dynamics on large time scales and dispersive effective eq., *SIAM J. Appl. Math.* 78(6), 2018.

A.Lamacz, B.S.: Representation of solutions to wave equations with profile functions, *Analysis & Appl.*, 2020.

# Stationary phase

Rough idea (no dispersion): Fourier-transform of ring function  $v^\varepsilon$

$$\begin{aligned}\hat{v}^\varepsilon(k, t) &\approx \int_{S^{d-1}} \int_0^\infty e^{-ik \cdot qr} \frac{r^{d-1}}{(ct)^{(d-1)/2}} V(r - ct, q, \varepsilon^2 t) dr dS(q) \\ &\approx \int_{S^{d-1}} \int_{\mathbb{R}} e^{-ik \cdot qct} e^{-ik \cdot qz} (ct)^{(d-1)/2} V(z, q, \varepsilon^2 t) dz dS(q) \\ &\approx \int_{S^{d-1}} e^{-ik \cdot qct} (ct)^{(d-1)/2} \hat{V}(k \cdot q, q, \varepsilon^2 t) dS(q) \\ &\approx \frac{1}{z_*} \int_{S^{d-1}} e^{-ik \cdot qct} (ct)^{(d-1)/2} |k \cdot q|^{(d-1)/2} \hat{u}_0(k \cdot q, q, \varepsilon^2 t) dS(q)\end{aligned}$$

with  $z_* := (i)^{(d-1)/2}$ . Use stationary phase for  $N = |k|ct$ . Recall  $\hat{u}^\varepsilon(k, t) \sim e^{i|k|ct}$ .

## Lemma (Stationary phase)

$d \in \{1, 2, 3\}$ ,  $\kappa \in S^{d-1}$ ,  $\phi \in C^1(S^{d-1}; \mathbb{R})$  supported in  $\{q \mid q \cdot \kappa \geq 0\}$ .

Then, as  $N \rightarrow \infty$ :

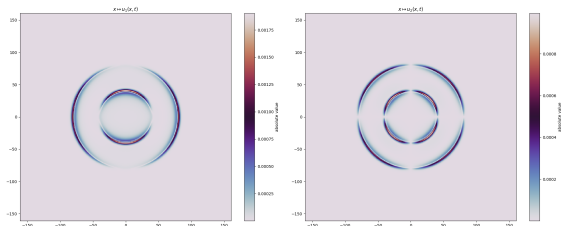
$$(2\pi i)^{-(d-1)/2} \int_{S^{d-1}} N^{(d-1)/2} e^{i(1-q \cdot \kappa)N} \phi(q) dS(q) \longrightarrow \phi(\kappa)$$

- Wave equation (continuous in space)
- Systems of wave equations (including elasticity)

## Results:

- Effective parameters from dispersion relation
- Multiple rings

**Example: Lamé system**  $\partial_t^2 u = 2\mu\Delta u + \lambda\nabla \operatorname{div} u$



Absolute values of  $u_1$  (left) and  $u_2$  (right) for fixed  $t_0$   
Pressure wave is ahead of shear wave

# Floquet-Bloch analysis for systems

The Floquet-Bloch transform of the solution is

$$(\mathcal{F}_{\text{FB}} u^\varepsilon)(x, k, t) = \sum_{m \in \mathbb{N}} b_m^\varepsilon(k, 0) \phi_m(x/\varepsilon, \varepsilon k) \exp(i\omega_m(\varepsilon k)t/\varepsilon)$$

Direction:  $\hat{k} = k/|k|$ . Define  $U^\varepsilon$  as the inverse Fourier transform of

$$\hat{U}^\varepsilon(k, t) = \sum_{m=1}^p B_m(k) \xi_m(\hat{k}) \exp(i\omega_m^*(\varepsilon k)t/\varepsilon)$$

- $p \in \mathbb{N}$ : There are  $p$  relevant Bloch eigenvalues
- $\xi_m \in \mathbb{C}^p$  are limits of Bloch functions,  $k \rightarrow 0$
- $p$  functions  $k \mapsto \omega_m^*(k)$  are approximate dispersion relations

$$\omega_m^*(\varepsilon k) = c_m(\hat{k})|\varepsilon k| + b_m(\hat{k})|\varepsilon k|^3$$

- Coefficients  $B_m(k)$  are projections of the initial data  $\hat{u}_0(k)$

Warning: This can be done only pointwise in the direction  $\hat{k}$

# Main result: Homogenization

Theorem 1 [Allaire, Lamacz-Keymling, B.S., 2026]

$$u_0 \in L^1(\mathbb{R}^d, \mathbb{C}^p) \cap L^2(\mathbb{R}^d, \mathbb{C}^p)$$

$\mathcal{F}(u_0)$  with compact support

$u^\varepsilon$  the solution to the oscillatory wave system

$T_0$  fixed and  $U^\varepsilon$  as above.

Then, as  $\varepsilon \rightarrow 0$ ,

$$\sup_{t \in [0, T_0/\varepsilon^2]} \|u^\varepsilon(\cdot, t) - U^\varepsilon(\cdot, t)\|_{L^2(\mathbb{R}^d)} \rightarrow 0$$

- Only  $m \in \{1, \dots, p\}$  are relevant
- Nyquist-Shannon to relate  $b_m^\varepsilon(k, 0)$  to  $\hat{u}_0(k)$
- Approximate the three factors

→ dependence of eigenvalues on parameter

## Description with rings, $d \leq 3$

- $\hat{G}_m$  from the representation of  $\hat{U}^\varepsilon(k, t)$
- Profile functions:

$$\hat{V}_m(\xi, q, \tau) := \frac{1}{z_*} |\xi|^{(d-1)/2} \hat{G}_m(\xi q, \tau)$$

- Single ring function:

$$w_m^\varepsilon(x, t) := \frac{1}{(c_m t)^{(d-1)/2}} V_m \left( |x| - c_m t, \frac{x}{|x|}, \varepsilon^2 t \right)$$

- Reconstruction with  $p$  rings:

$$w^\varepsilon(x, t) := \sum_{m=1}^p w_m^\varepsilon(x, t)$$

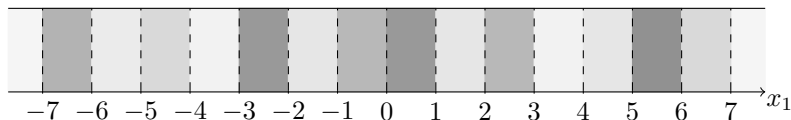
Theorem 2 [Allaire, Lamacz-Keymling, B.S., 2026]

For every sequence  $t_\varepsilon$  with  $\varepsilon^2 t_\varepsilon \rightarrow \tau > 0$  holds

$$\|w^\varepsilon(\cdot, t_\varepsilon) - U^\varepsilon(\cdot, t_\varepsilon)\|_{L^2(\mathbb{R}^d)} \rightarrow 0$$

# Stochastic medium. Example: i.i.d. medium

Let  $(a_j)_{j \in \mathbb{Z}}$  and  $(\rho_j)_{j \in \mathbb{Z}}$  be i.i.d. random variables, e.g.: uniform in  $[1, 2]$



Set  $a(x) = a_j$  and  $\rho(x) = \rho_j$  for  $x \in [j, j + 1)$ . Then  $a_\varepsilon(x) := a(x/\varepsilon)$

Effective coefficients  $\bar{\rho} = \langle \rho_0 \rangle$  and  $\bar{a} := \langle a_0^{-1} \rangle^{-1}$

**Task:** Compare solutions on time intervals  $[0, \tau\varepsilon^{-\beta}]$

$$u^\varepsilon : \mathbb{R} \times [0, \infty) \times \Omega_{\mathcal{P}} \rightarrow \mathbb{R}$$

$$\rho_\varepsilon \partial_t^2 u^\varepsilon - \partial_x (a_\varepsilon \partial_x u^\varepsilon) = f$$

$$\bar{u} : \mathbb{R} \times [0, \infty) \rightarrow \mathbb{R}$$

$$\bar{\rho} \partial_t^2 \bar{u} - \partial_x (\bar{a} \partial_x \bar{u}) = f$$

Both with trivial initial conditions, e.g.:  $u^\varepsilon(\cdot, 0) = \partial_t u^\varepsilon(\cdot, 0) = 0$

# Time horizon

Always:  $f \in C^2(\mathbb{R} \times \mathbb{R}_+, \mathbb{R})$  with compact support and  $T_0 > 0$  arbitrary

We say **homogenization works** for  $\beta \geq 0$  if

$$\lim_{\varepsilon \rightarrow 0} \sup_{t \in [0, T_0 \varepsilon^{-\beta}]} \mathbb{E} \left( \|\partial_t u^\varepsilon(\cdot, t) - \partial_t \bar{u}(\cdot, t)\|_{L^2(\mathbb{R})}^2 \right) = 0$$

## Theorem (Critical parameters)

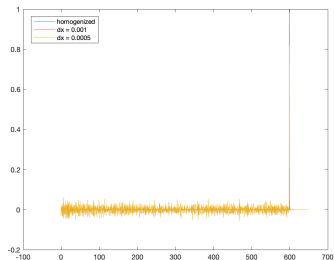
- 1 For  $\beta < \beta_- := \frac{1-\gamma}{1+\gamma}$ : For all coefficients  $(\rho, a)$  of class  $\gamma$ ,  
**homogenization works** for  $\beta$
- 2 For  $\beta > \beta_+ := \frac{1-\gamma}{\gamma}$ : There exist coefficients  $(\rho, a)$  of class  $\gamma$   
such that **homogenization does not work** for  $\beta$

**Open problem:** Consider i.i.d. model with  $\gamma = 1/2$

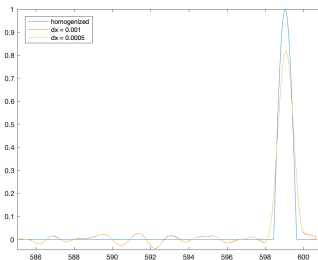
Homogenization works for  $\beta < \beta_- = \frac{1}{3}$ , it does not work for  $\beta > \beta_+ = 1$

**What is the critical parameter?**

# Numerical tests



Solution on  $x \in (0, 700)$



Zoom

Numerical results for  $\varepsilon = 0.01$  and  $t_0 = 600$

Calculations with discretization  $dx = 0.001$  and with  $dx = 0.0005$

Complexity:  $10^6$  unknowns, time resolution adds a second dimension that requires the same number of points

- **Long time homogenization:** Homogenization on time intervals  $[0, T_0/\varepsilon^2]$   $\rightarrow$  dispersive equations
- **Ring solutions:** Solutions are superpositions of rings
- **Profile:** Slow profile evolution equation:  $\partial_\tau V = b \partial_z^3 V$
- **Stochastic homogenization:** Large time homogenization in 1D

## Thank you!

- [1] A.Lamacz: Dispersive effective models for waves in heterogeneous media. *Math. Models Methods Appl. Sci.* 21(9), 2011
- [2] T.Dohnal, A.Lamacz, B.S.: Bloch-wave homogenization on large time scales and dispersive effective wave equations. *Multiscale Model. Simul.* 12(2), 2014
- [3] T.Dohnal, A.Lamacz, B.S.: Dispersive homogenized models and coefficient formulas for waves in general periodic media. *Asymptot. Anal.* 93(1-2), 2015
- [4] B.S., F.Theil: Lattice dynamics on large time scales and dispersive effective equations, *SIAM J. Appl. Math.* 78(6), 2018
- [5] A.Lamacz, B.S.: Representation of solutions to wave equations with profile functions, *Analysis and Applications*, 2020
- [6] M.Schäffner and B.S., The time horizon for stochastic homogenization of the one-dimensional wave equation, *Asymptotic Analysis*, 2024