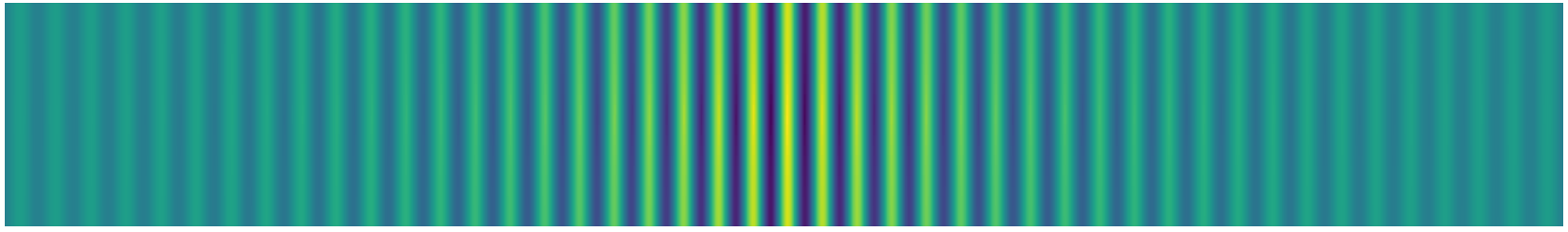
Edge-modes at interfaces between periodic media

Ben Schweizer (TU Dortmund)



Bonn, 27. October 2026

Three talks in one ...

- Mathematical theorem: Existence of an eigenvalue
- Physics and intuition
- Tool: Floquet-Bloch (student-friendly)

Mathematical setting

Waveguide geometry

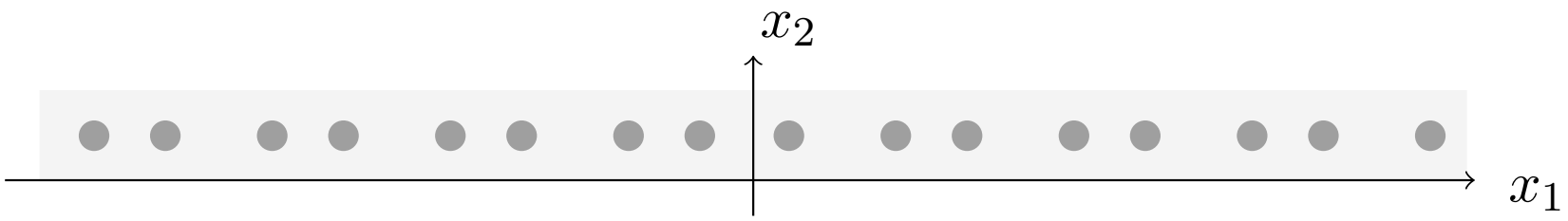
$$\Omega = \mathbb{R} \times \Sigma \subset \mathbb{R}^d$$

Cross-section $\Sigma \subset \mathbb{R}^{d-1}$ is a bounded Lipschitz-domain, $d \geq 2$

$$-\nabla \cdot (a \nabla u) = \lambda\, b\, u$$

(homogeneous Neumann)

Coefficients $a, b : \Omega \to \mathbb{R}$ with positive lower bounds
Solutions $u : \Omega \to \mathbb{C}$ and $\lambda \in \mathbb{R}$ a spectral parameter



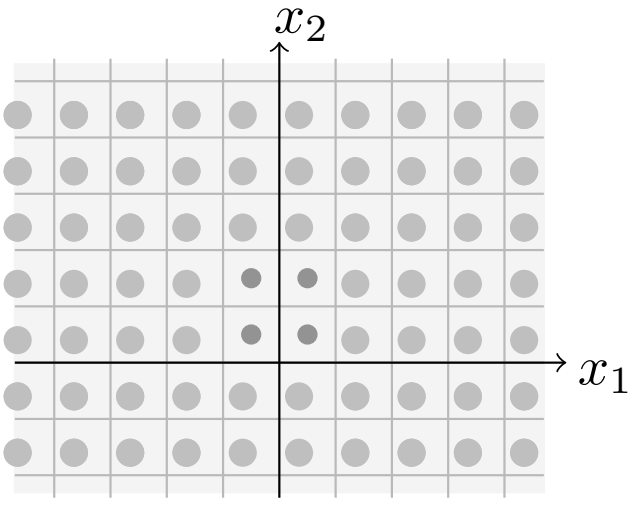
$\longrightarrow$ **We are interested in spectral properties**

Important special case: Periodic media

Both a and b are 1-periodic in $x_1 \in \mathbb{R}$

Related problems

Locally perturbed periodic medium



- **Existence:** Essentially as in unperturbed case

  A. Kirsch and many others

- **Spectral properties:** As in unperturbed case, possibly additional localized modes

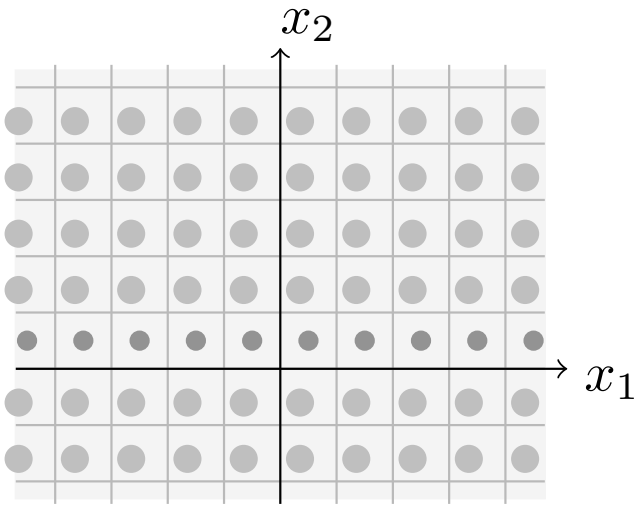
- **Homogenization:** Study, for $p \in \mathbb{R}^d$, the corrector $w_p : \mathbb{R}^d \to \mathbb{R}$

$$-\nabla \cdot (a(x/\varepsilon)(p + \nabla w_p)) = 0\,, \qquad a = a^{\mathrm{per}} + \tilde{a}\,, \qquad \tilde{a} \in L^r(\mathbb{R}^d)$$

X. Blanc, C. Le Bris, P.-L. Lions, *Comm.PDE,* 2018

Related problems

Periodic medium, perturbed along a manifold



- **Spectral properties:** Spectral gaps of the medium are smaller

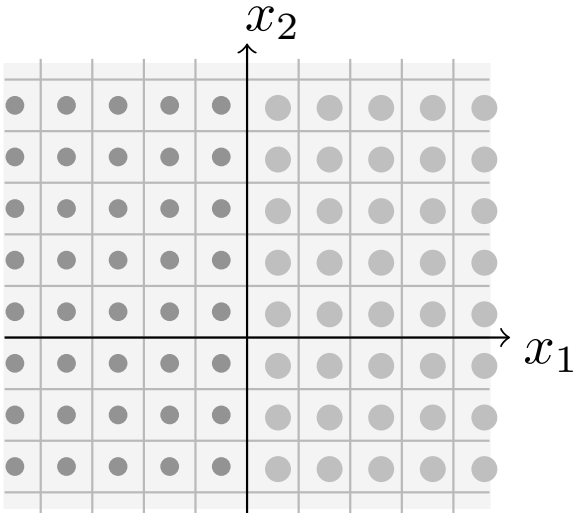
$\longrightarrow$ *modes can travel along the defect*

B. Brown, V. Hoang, M. Plum, and I. Wood, *Math.Nachr.*, 2014

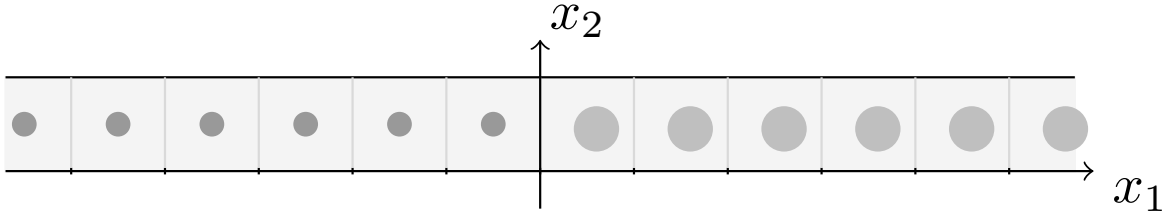
Maxwell: A. Kirsch and B.S., *IMA J.Appl.Math.*, 2025

Related problems

Two different periodic media combined (our case!)



Restrict to waveguides



- **Existence:** Results are qualitatively similar to the periodic case

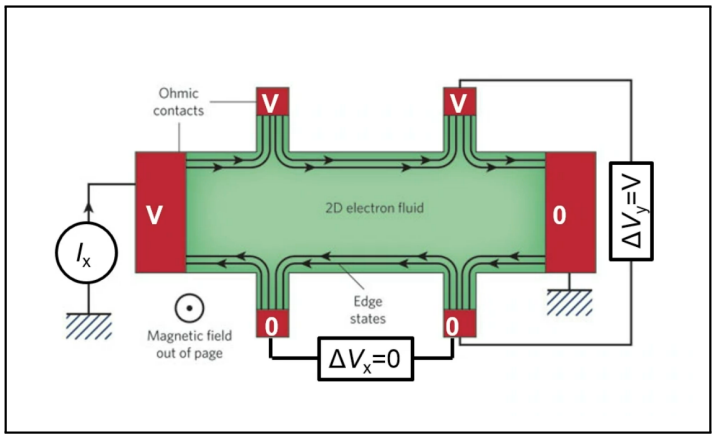
e.g.: B.S., *Eur.J.Appl.Math.*, 2021

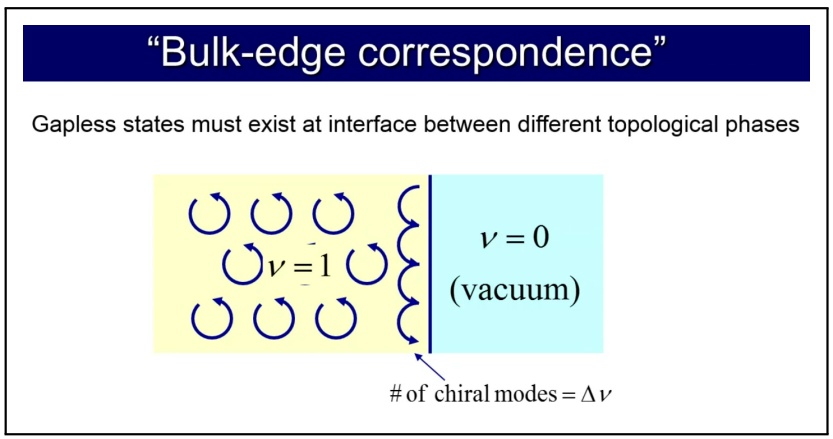
- **Spectral properties?** **Topic of this talk!**

Integer quantum Hall effect

K. v. Klitzing 1980
Nobel Prize 1985

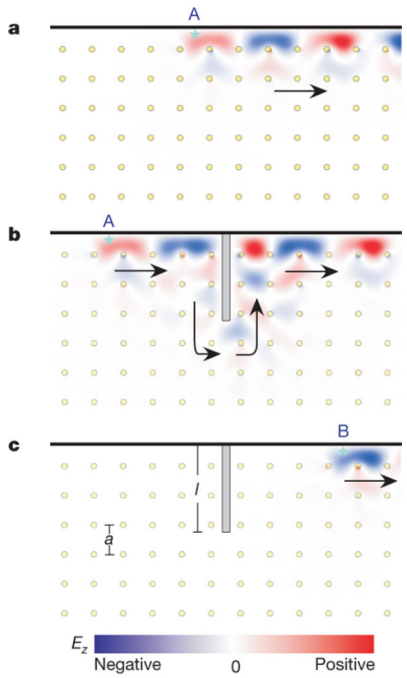
Prof. M. Fuhrer (2016 on YouTube)
"The 2016 Nobel Prize in Physics"
Main contribution: TKNN-paper of 1982





Numerical experiments

Nature, 2009



Z Wang *et al. Nature* **461**, 772-775 (2009) doi:10.1038/nature08293

Spectral results for combined media (selection)

- C. Fefferman, J. Lee-Thorp, M. Weinstein, Topologically **protected states** in one-dimensional systems, *Comm.PDE,* 2016
- A. Drouot, C. Fefferman, M. Weinstein, Defect modes for dislocated periodic media, *Comm.Math.Phys.,* 2020

  Both 1D with adiabatic "sign"-function κ. Unbounded Dirac operator

$$\not{D}_* = \begin{bmatrix} 0 & \nu_* k' \\ \nu_* k' & 0 \end{bmatrix} D_t + \vartheta_* \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \kappa : L^2(\mathbb{R}, \mathbb{C}^2) \to L^2(\mathbb{R}, \mathbb{C}^2)$$

  $\longrightarrow$ closely related to our reduced spatial dynamics

- A. Drouot, The **bulk-edge correspondence** for continuous honeycomb lattices, *Comm.PDE,* 2019

  Thm. 1: $c_1(\mathcal{E}_{\delta,\pm}) = \mp \mathbf{sign}(\theta_*)$ *Thm. 2:* $\mathcal{N} = c_1(\mathcal{E}_{\delta,+}) - c_1(\mathcal{E}_{\delta,-})$

- G. Bal, Continuous bulk and interface description of **topological insulators,** *J.Math.Phys.,* 2019
- A. Drouot, **Microlocal analysis** of the bulk-edge correspondence, *Comm.Math.Phys.,* 2021

  last two in the spirit of the above Thm. 2

- B. Delourme, S. Fliss, P. Joly, E. Vasilevskaya. Trapped modes in thin and infinite ladder like domains, *Asympt. Anal.,* 2020

Periodic media: Floquet-Bloch transform I

Waveguide $\Omega = \mathbb{R} \times \Sigma$ with periodicity cell $W = (0,1) \times \Sigma$

The Floquet-parameter is $k \in I = [-\pi, \pi]$

Think of solutions $u(x) = e^{ikx}$

Definition

The Floquet-Bloch transform is a bounded linear map

$$\mathcal{F}_{\mathrm{FB}} : L^2(\Omega) \to L^2\big(W \times I\big)\,, \qquad u \mapsto \hat{u}$$

A function $u \in C_c^\infty(\Omega)$ is mapped onto

$$\hat{u}(x,k) \;:=\; \frac{1}{\sqrt{2\pi}} \sum_{\ell \in \mathbb{Z}} u(x + \ell\, e_1)\, e^{-i\ell k}$$

for every $x \in W$ und $k \in I$

First question: Does $\hat{u}$ contain all information about u?

Periodic media: Floquet-Bloch transform II

Lemma

$\mathcal{F}_{\mathrm{FB}} : L^2(\Omega) \to L^2\big(W \times I\big)$ is a unitary isometry.
The inverse is, for almost every $x \in W$ and every $\ell \in \mathbb{Z}$:

$$u(x + \ell\, e_1) = \frac{1}{\sqrt{2\pi}} \int_I \hat{u}(x,k)\, e^{i\ell k}\, dk$$

Central part is this easy calculation:

$$\begin{aligned}
\langle \hat{u}, \hat{v} \rangle_{L^2(W\times I)} &= \int_I \int_W \hat{u}(x,k)\, \overline{\hat{v}(x,k)}\, dx\, dk \\
&= \frac{1}{2\pi} \int_I \int_W \sum_{\ell,j\in\mathbb{Z}} u(x+\ell\, e_1)\, \overline{v(x+j\, e_1)}\, e^{-i(\ell-j)k}\, dx\, dk \\
&= \int_W \sum_{\ell\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} \delta_{j,\ell}\; u(x+\ell\, e_1)\, \overline{v(x+j\, e_1)}\, dx \\
&= \int_W \sum_{\ell\in\mathbb{Z}} u(x+\ell\, e_1)\, \overline{v(x+\ell\, e_1)}\, dx = \int_{\mathbb{R}\times\Sigma} u\, \bar{v} = \langle u, v \rangle_{L^2(\Omega)}
\end{aligned}$$

Periodic media: Floquet-Bloch transform III

Recall: $\Omega = \mathbb{R} \times \Sigma$, $I = [-\pi, \pi]$, $W = (0,1) \times \Sigma$

Quasiperiodic function

For $k \in I$, we say that $u : \Omega \to \mathbb{C}$ is k-quasiperiodic iff

$$u(x + e_1) = e^{ik}\, u(x)$$

$u_k \in H^1_k(W)$ means: The k-quasiperiodic extension is $H^1_{\mathrm{loc}}(\Omega)$

Quasiperiodic problem

Given $k \in I$ and $f_k \in L^2(W)$, seek $u_k \in H^1_k(W)$ such that:

$$\int_W \{a\, \nabla u_k \cdot \nabla \bar{\varphi} - \lambda\, b\, u_k\, \bar{\varphi}\} = \int_W f_k\, \bar{\varphi} \qquad \forall \varphi \in H^1_k(W)$$

Very helpful! Spectral problem on a bounded domain

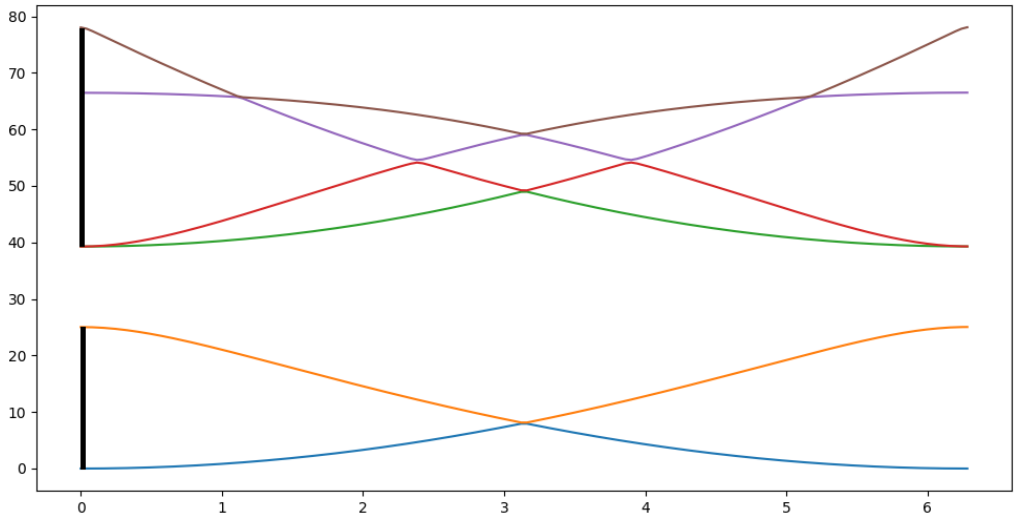
Periodic media: Floquet-Bloch transform IV

Here: Coefficients a and b are 1-periodic in x_1

Theorem (Characterization of the spectrum)

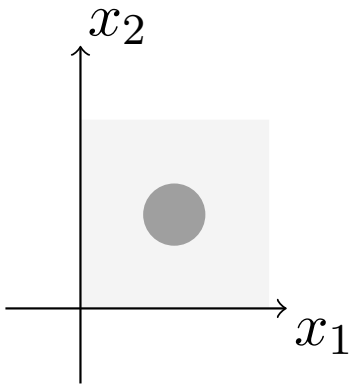
$\lambda \in \mathbb{C}$ *is in the spectrum of the* Ω*-problem* **iff,**

$\exists k \in I:$ λ *is an eigenvalue of the* k*-quasiperiodic problem in* W

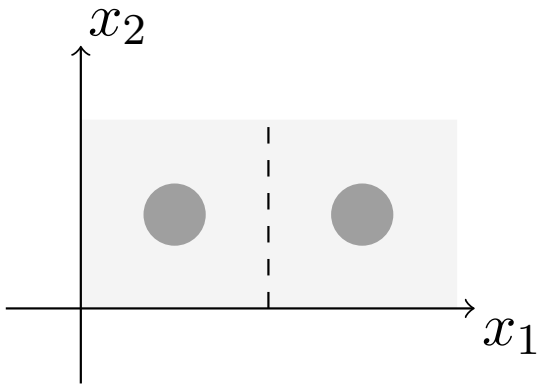


Typical eigenvalue curves for Floquet-parameters $k \in \mathbb{R}$

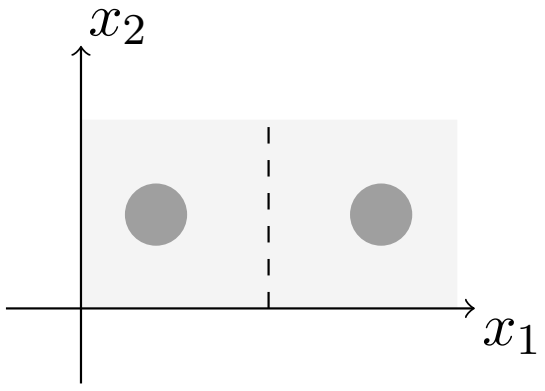
Periodic media: Symmetry



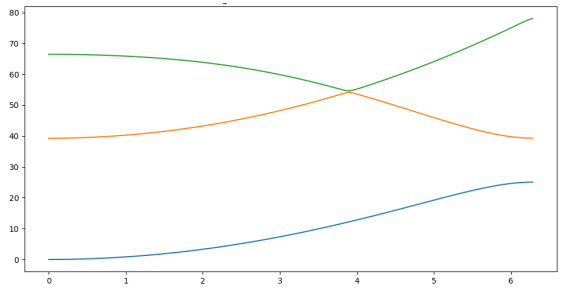
$\frac{1}{2}$-periodicity cell

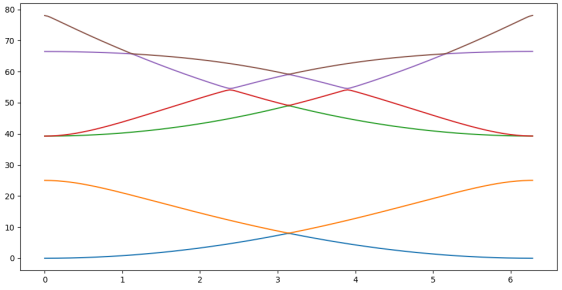


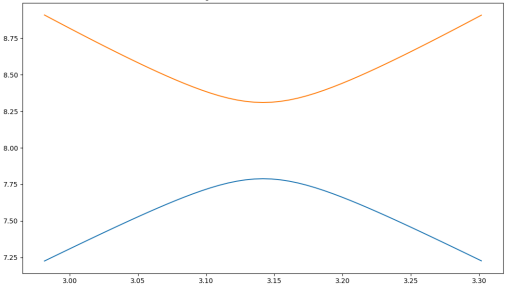
interpreted with
1-periodicity cell



A perturbed
1-periodic medium







Dispersion curves (= First eigenvalues of quasiperiodic problems)

Observations:

Appearance of a
Dirac-point

New spectral gap

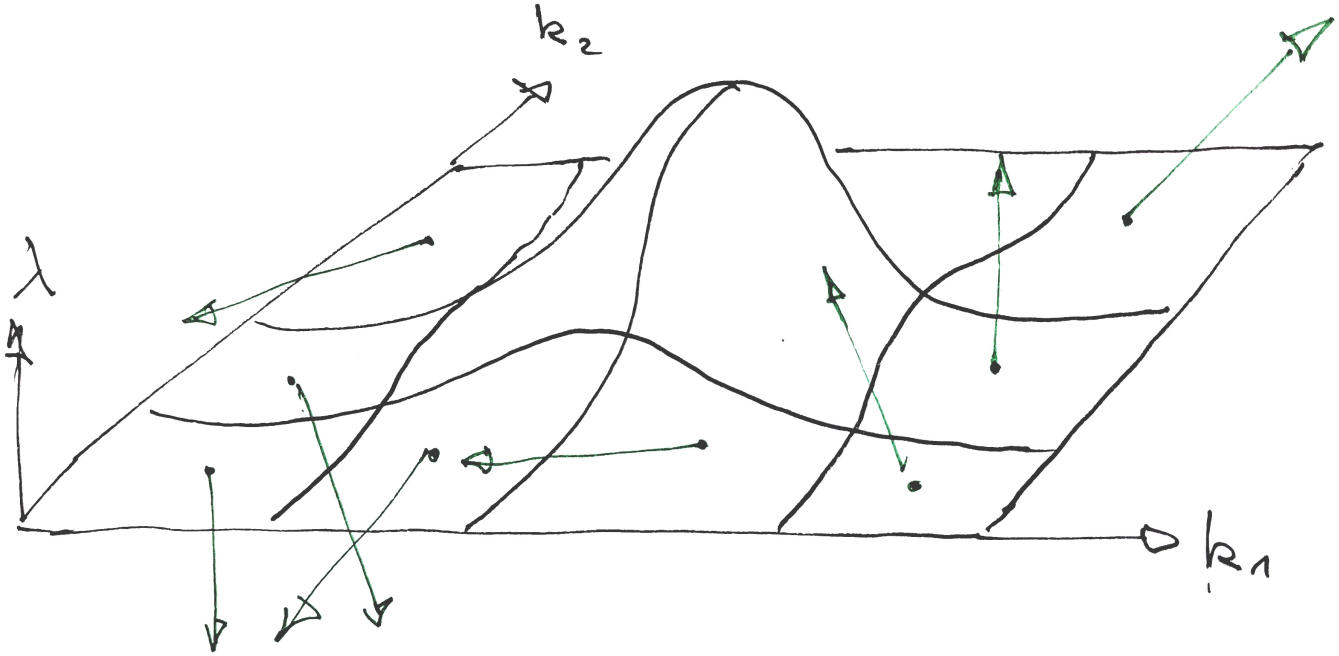
Periodic media: Some topology (smart version)

Consider $d = 2$ and the Brillouin-zone $B := [-\pi, \pi]^2$

Floquet-parameter $k = (k_1, k_2) \in B$ and $m \in \mathbb{N}$ an index

$u_m : B \to L^2(W, \mathbb{C}) =: X, \quad k \mapsto u_m(\,\cdot\,; k)$ (assume simplicity)

Eigenvalue surface below λ_0



$\longrightarrow$ **complex line bundle over torus** $\mathbb{T}^2$

First Chern number of this bundle is a topological invariant

Periodic media: Some topology (less smart version)

Brillouin-zone $k \in B := [-\pi, \pi]^2$ (not the torus)

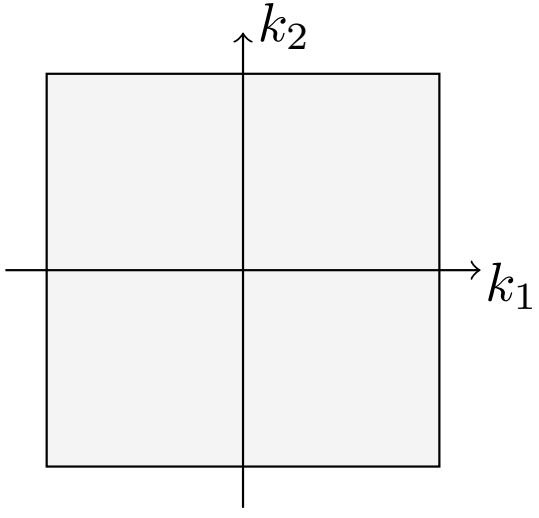
$u_m : B \to X = L^2(W, \mathbb{C}), \quad k \mapsto u_m(\,\cdot\,; k)$ continuous

Construct phase factor $\varphi : \partial B \to \mathbb{R}$

continuous on $\partial B \setminus \{(\pi, \pi)\}$

"correcting phases" in the sense that

$u_m(k)e^{i\varphi(k)}$ is periodic on ∂B



Topological invariant:

φ jumps in upper right corner by $2\pi z$, some $z \in \mathbb{Z}$

(z uniquely defined)

$\longrightarrow$ Chern-number

Media for interface modes

Unperturbed problem in Dirac point $k = \pi$ (use $e^{ik} = -1$)

Find $\Phi : (0,1) \times \Sigma \to \mathbb{C}$ solving

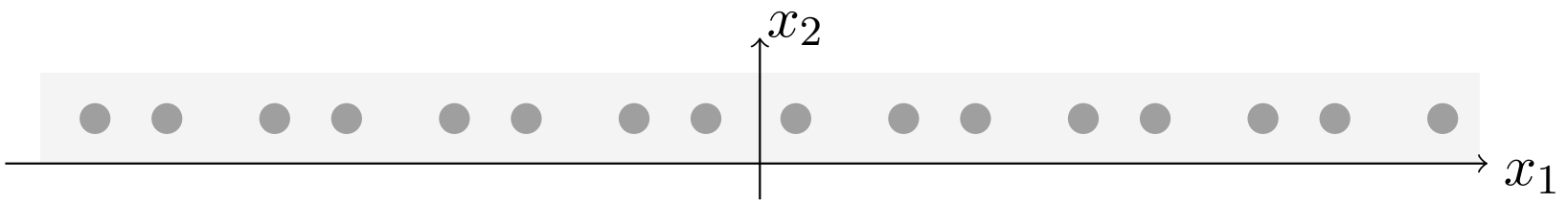
$$-\nabla \cdot (a \nabla \Phi) = \lambda\, b\, \Phi$$
$$\Phi|_{x_1=1} = -\Phi|_{x_1=0}\,, \quad \partial_1 \Phi|_{x_1=1} = -\partial_1 \Phi|_{x_1=0}$$

Two-dimensional solution space, spanned by V and W

Intuition: $V(x) \approx e^{i\pi x_1}$ *and* $W(x) \approx e^{-i\pi x_1}$

Perturbation $\longrightarrow$ spectral gap

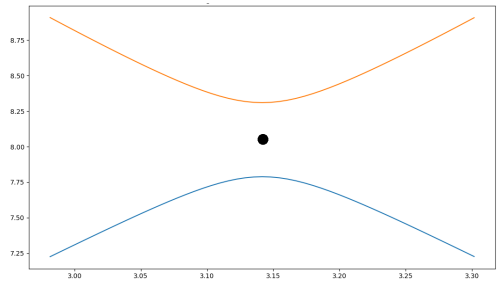
Use opposite perturbations left and right



Main result

Let λ_0 be the Dirac value
For $\mu_0 > 0$ consider

$$\Lambda_\varepsilon := (\lambda_0 - \varepsilon\mu_0, \lambda_0 + \varepsilon\mu_0) \subset \mathbb{R}$$



Theorem (Existence of an interface mode)

$(a_\varepsilon, b_\varepsilon)$: coefficients of the combined medium as above, $q \neq 0$

Assume: Λ_ε contained in spectral gap for all ε

Then: $\exists\, \varepsilon_0 > 0\ \forall \varepsilon \in (0, \varepsilon_0)$: problem in combined medium has an eigenvalue in Λ_ε with eigenfunction in $L^2(\mathbb{R} \times \Sigma)$

- Proof uses homogenization $\varepsilon \to 0$
- Note: In this limit, the spectral gap closes
- The eigenfunction is an interface mode

For the proof: Reduced spatial dynamics

Coefficients (a_0, b_0) — perturbation $\longrightarrow$ $(a_\varepsilon, b_\varepsilon)$

Inhomogeneous ε-problem

$$-\nabla \cdot (a_\varepsilon \nabla u) - \lambda\, b_\varepsilon\, u = f$$

Special f: For an envelope function $g : \mathbb{R} \to \mathbb{C}$, set

$$f(x) := V(x) g(x_1)$$

Ansatz for the solution $u = u^\varepsilon$ in $\Omega = \mathbb{R} \times \Sigma$:

$$u(x) \approx \alpha(x_1) V(x) + \beta(x_1) W(x)$$

Unknowns: $\alpha, \beta : \mathbb{R} \to \mathbb{C}$ (envelopes)

Homogenization $\longrightarrow$ effective system

For parameters $A, p \in \mathbb{R}$ and $q \in \mathbb{C}$ (homogenized coefficients)

$$\begin{aligned} \alpha' &= iAg + ip\alpha + iq\beta \\ \beta' &= \quad\; - ip\beta - i\bar{q}\alpha \end{aligned}$$

Why this reduced spatial dynamics?

The equation for $u \approx \alpha V + \beta W$, tested with ψV, is

$$\int_\Omega a\nabla(\alpha V + \beta W)\cdot\nabla(\psi\bar{V}) - \lambda\int_\Omega b(\alpha V + \beta W)\,(\psi\bar{V}) \approx \int_\Omega g\psi\, V\bar{V}$$

Expanding terms with product rule yields

$$\int_\Omega a\left\{\alpha\psi|\nabla V|^2 + \beta\psi(\nabla W\cdot\nabla\bar{V})\right\}$$

$$+\int_\Omega a\left\{\alpha'\psi V\partial_1\bar{V} + \alpha\psi'\partial_1 V\bar{V} + \beta'\psi W\partial_1\bar{V} + \beta\psi'\partial_1 W\bar{V}\right\}$$

$$+\int_\Omega a\left\{\alpha'\psi'|V|^2 + \beta'\psi' W\bar{V}\right\}$$

$$-\lambda\int_\Omega\left\{b\alpha\psi|V|^2 + b\beta\psi W\bar{V}\right\} \approx \int_\Omega g\psi\,|V|^2$$

- Red terms give the desired terms α' and β'
- Black terms form the right-hand side
- Blue terms are small (two derivatives)

$$\alpha' = iAg + ip\alpha + iq\beta$$
$$\beta' = \quad - ip\beta - i\bar{q}\alpha$$

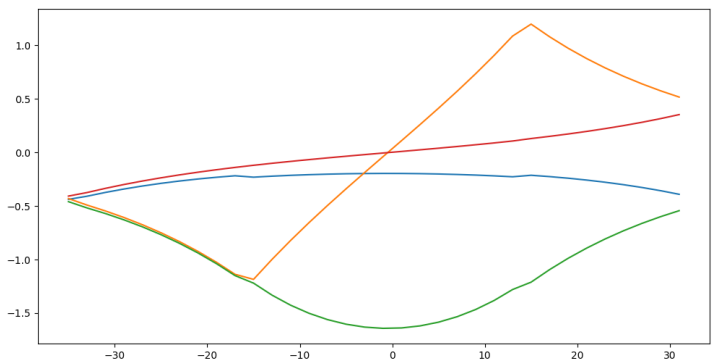
Numerical investigation of reduced spatial dynamics

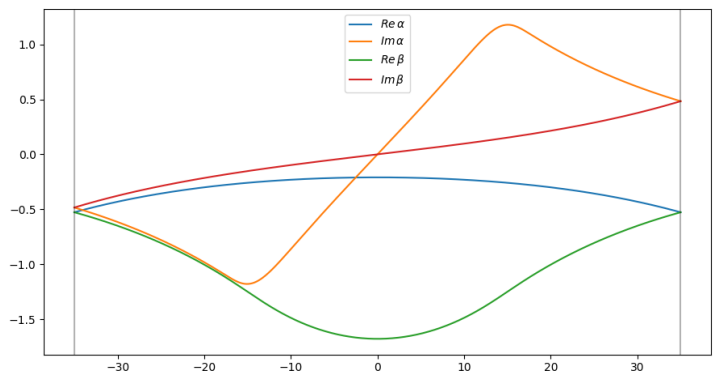
Left Figure: Solve, for $f(x) := V(x)g(x_1)$,

$$-\nabla \cdot (a_\varepsilon \nabla u) - \lambda\, b_\varepsilon\, u \;=\; f$$

Use numerical **projections** at position $r \in \mathbb{Z}$:

$$\alpha_{\mathrm{num}}(r) = \int_{Q_r} u\bar{V} \quad \text{and} \quad \beta_{\mathrm{num}}(r) = \int_{Q_r} u\bar{W}$$





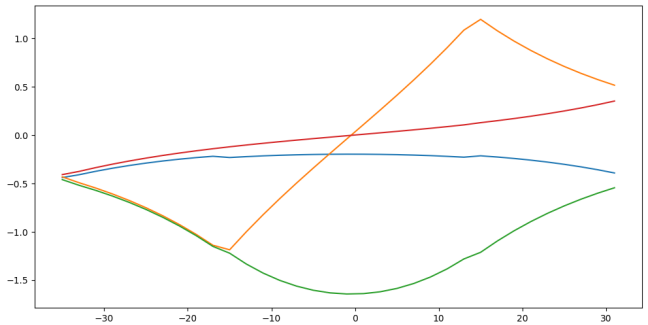
Right Figure: Solutions of

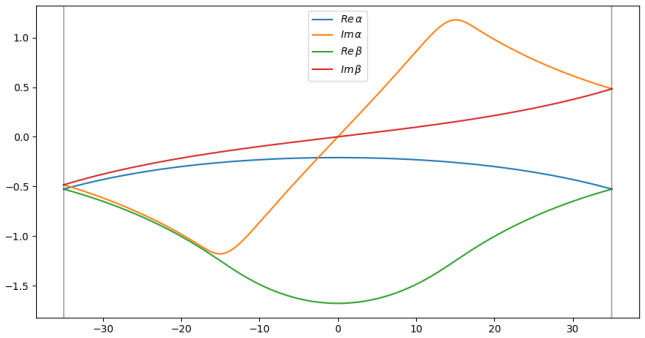
$$\alpha' = iAg + ip\alpha + iq\beta\,, \qquad \beta' = -ip\beta - i\bar{q}\alpha$$

(for $A = 1/(2\pi)$, $p = 0$, $q = 0.045$ and b.c. $\alpha(\pm 35) = \beta(\pm 35)$

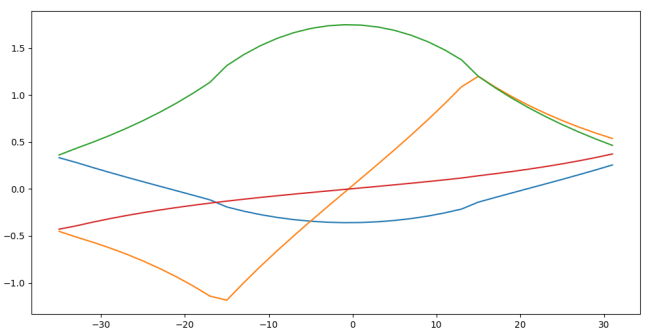
Numerical investigation of reduced spatial dynamics

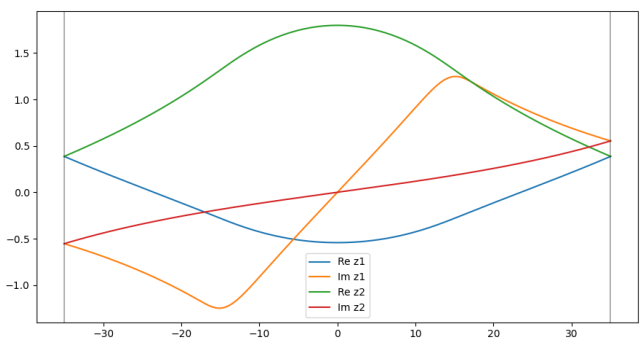
Medium I (in mixed medium used for $x_1 > 0$), $q = q_0$



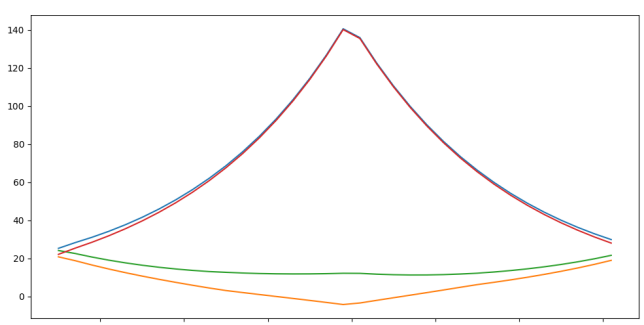


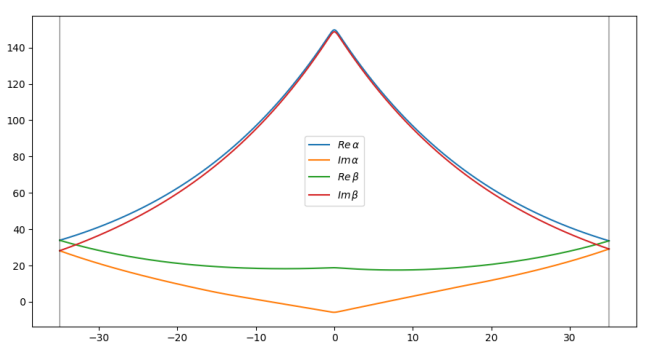
Medium II, $q = -q_0$



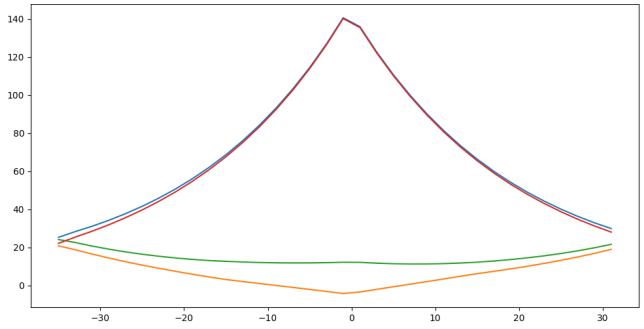


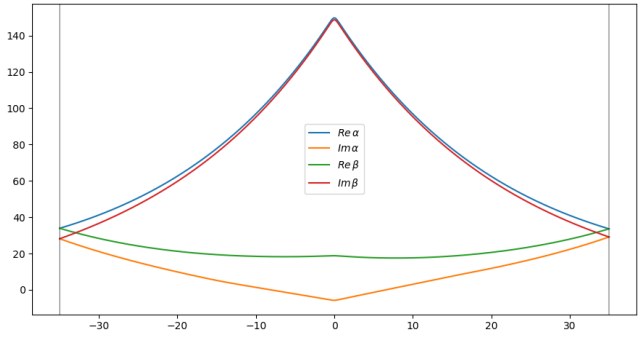
Mixed medium, I right, II left





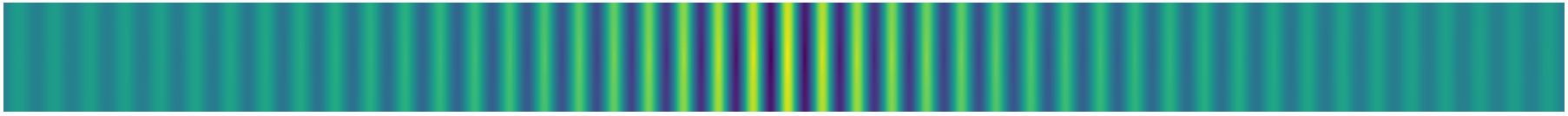
Conclusions





Proof of the main result: Mixed medium, fixed right-hand side g, solution is very large $\Rightarrow$ Existence of an eigenvalue

- **Spectrum:** Within the spectral gap of the single media, the combined medium has an eigenvalue
- **Interface mode:** The eigenmode is localized at $x_1 = 0$



Thank you!