Porous Media and Plasticity - Homogenization for Equations with Hysteresis

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Modelling subsurface flow

Describe the flow of water in unsaturated porous media

Variables

domain	$\Omega \subset \mathbb{R}^N$
saturation	$s : \Omega \times (0,T) \to \mathbb{R}$
pressure	$p : \Omega \times (0,T) \to \mathbb{R}$
velocity	$v : \Omega \times (0,T) \to \mathbb{R}^N$

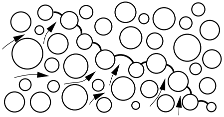
Equations

Darcy law	$v = -k(s)\nabla p$
conservation law	$\partial_t s + \nabla \cdot v = 0$
some relation between	p and s

We combine these to the evolution equation

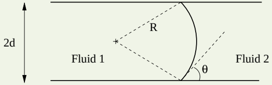
$$\partial_t s = \nabla \cdot (k(s)\nabla p).$$

Microscopic Analysis I



relation $p \leftrightarrow s$ depends on pores

Tube-Model



d radius of the tube

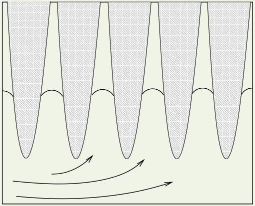
θ contact angle

$\mathcal{H} = R^{-1}$ curvature

β surface tension

$p = \beta\mathcal{H} = F(d)$

Variable radius



At a given saturation s, pores of radius $d_0(s)$ must be filled.

This needs the pressure

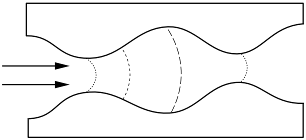
$$p = p_c(s)$$

Richards equation:

$$\partial_t s = \nabla \cdot (k(s)\nabla[p_c(s)])$$

Microscopic Analysis II: Play-type capillary hysteresis

In reality, the radius of the tubes is oscillatory.



This implies that an interval of pressures is allowed for one saturation,

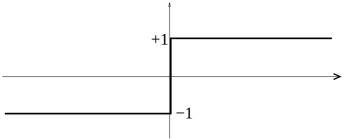
$$p \in [p_1, p_2] =: p_{c,0}(s) + [-\gamma, \gamma].$$

with the rule: upper/lower value for increasing/decreasing saturation

Resulting model

$$p \in p_{c,0}(s) + \gamma\ \mathbf{sign}(\partial_t s)$$

with the multi-valued sign-function, $\mathbf{sign}(0) = [-1, 1]$.

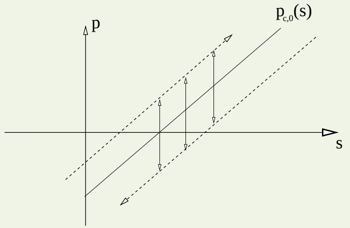


- S., A stochastic model for fronts in porous media, Ann. Mat. Pura Appl. 2005
- S., Laws for the capillary pressure ..., SIAM J. Math. Analysis, 2005

Existence results for the hysteresis model

$$p \in p_{c,0}(s) + \gamma\, \mathbf{sign}(\partial_t s)$$

Hysteresis relation p to s



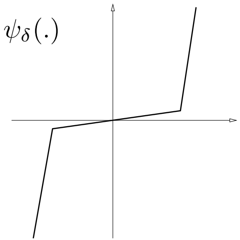
Theorem

Let $T > 0$ and let initial data s_0 be compatible. Then there exists a weak solution of the differential inclusion $\partial_t s = \Delta p$ with $p \in as + b + \gamma\, \mathbf{sign}(\partial_t s)$

Method of proof: Regularization.

$$\text{Discretize } \Omega \quad \longrightarrow \quad h > 0$$

$$\text{Approximate } \psi = \mathbf{sign}^{-1} \quad \longrightarrow \quad \delta > 0$$



$$\partial_t s^{h,\delta} = \psi_\delta([p^{h,\delta} - as^{h,\delta} - b]/\gamma)$$

$$\Delta_{(h)} p^{h,\delta} = \psi_\delta([p^{h,\delta} - as^{h,\delta} - b]/\gamma)$$

Plasticity equations

Variables

domain	$\Omega \subset \mathbb{R}^N$
displacement	$u : \Omega \times (0,T) \to \mathbb{R}^N$
strain	$\epsilon : \Omega \times (0,T) \to \mathbb{R}^{N\times N}$
stress	$\sigma : \Omega \times (0,T) \to \mathbb{R}^{N\times N}$

Equations

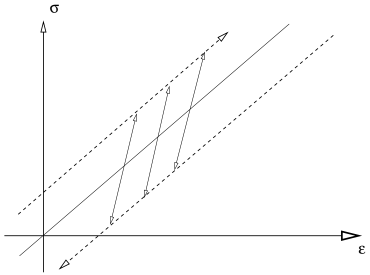
conservation law	$\rho\, \partial_t^2 u = \nabla \cdot \sigma + f$
strain relation	$\epsilon = \frac{1}{2}(\nabla u + (\nabla u)^\perp)$
a relation between	ϵ and σ

Linear elasticity

One uses the simplest choice,

$$\sigma = A \cdot \epsilon.$$

The observation in **plasticity** is: the material flowing beyond some stress



Melan-Prager

One-dimensional relation

$$\alpha\epsilon \in \sigma - \gamma\, \mathbf{sign}(\partial_t \epsilon - \beta \partial_t \sigma).$$

Hysteresis problems of plasticity and hydrology

Wave equation with hysteresis

$$\begin{aligned} \rho\partial_t^2 u &= \partial_x \sigma + f \\ \partial_x u &= \epsilon + \beta\sigma \\ \alpha\epsilon &\in \kappa\sigma - \gamma\,\mathbf{sign}(\partial_t \epsilon). \end{aligned}$$

$\alpha, \beta, \gamma, \kappa$ are parameters.

Richards equation with hysteresis

$$\begin{aligned} \partial_t s &= \Delta p \\ p &\in as + b + \gamma\,\mathbf{sign}(\partial_t s) \end{aligned}$$

a, b, γ are parameters.

Energy estimate, Plasticity. Testing with $\partial_t u$ gives, for $f = 0$

$$\begin{aligned} \partial_t \frac{1}{2}\int \rho|\partial_t u|^2 &= -\int \sigma\,\partial_t\partial_x u = -\int \sigma\,\partial_t(\epsilon + \beta\sigma) \\ &\in -\partial_t \frac{1}{2}\int \beta|\sigma|^2 - \int \left[\frac{\alpha}{\kappa}\epsilon + \frac{\gamma}{\kappa}\mathbf{sign}(\partial_t\epsilon)\right]\partial_t\epsilon \\ &= -\partial_t \frac{1}{2}\int \beta|\sigma|^2 - \partial_t \frac{1}{2}\int \frac{\alpha}{\kappa}|\epsilon|^2 - \int \frac{\gamma}{\kappa}|\partial_t\epsilon| \end{aligned}$$

Hysteresis problems of plasticity and hydrology

Wave equation with hysteresis

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$\alpha, \beta, \gamma, \kappa$ are parameters.

Richards equation with hysteresis

$$\begin{aligned} \partial_t s &= \Delta p \\ p &\in as + b + \gamma\,\mathbf{sign}(\partial_t s) \end{aligned}$$

a, b, γ are parameters.

Energy estimate, Richards. Multiplication of the first equation with p and integration over Ω gives

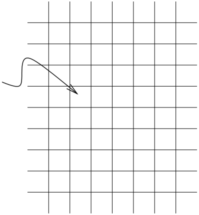
$$\begin{aligned} \int |\nabla p|^2 &= -\int p\,\partial_t s \in \int [as + b + \gamma\,\mathbf{sign}(\partial_t s)]\partial_t s \\ &= \partial_t \int \left\{ \frac{a}{2}|s|^2 + bs \right\} + \int \gamma |\partial_t s|. \end{aligned}$$

The fundamental question

We are interested in composite materials (periodic or stochastic).

The material parameters are constant in each cell, chosen randomly in each cell.

The grid-spacing is $\varepsilon > 0$.



Fundamental question of homogenization

If u^ε are solutions to the ε-problems, and $u^\varepsilon \rightharpoonup u^*$.
What is the equation for u^*?

- Two-scale convergence (Allaire, ...)
- Energy method (Tartar, ...)

B.S. Homogenization of the Prager model in one-dimensional plasticity. Continuum Mechanics and Thermodynamics 20(8), 2009.

B.S. Averaging of flows with capillary hysteresis in stochastic porous media. European Journal of Applied Mathematics 18, 2007.

Main result on plasticity

Let u^ε be a solution to the problem with oscillating parameters,

$$\begin{aligned}
\partial_t^2 u^\varepsilon &= \partial_x \sigma^\varepsilon + f \\
\partial_x u^\varepsilon &= \epsilon^\varepsilon + \beta^\varepsilon \sigma^\varepsilon \\
\alpha^\varepsilon \epsilon^\varepsilon &\in \kappa^\varepsilon \sigma^\varepsilon - \gamma^\varepsilon \mathbf{sign}(\partial_t \epsilon^\varepsilon).
\end{aligned}$$

Idea: Material label $y \in I := [0,1]$. The measure $\mu \in \mathcal{M}(I)$ denotes the probability distribution of materials.
The strain in material $y \in I$ is $w(x,t,y)$. **Problem (P_*) is**

$$\begin{aligned}
\partial_t^2 u^* &= \partial_x \sigma^* + f \\
\partial_x u^* &= \int_I w(y)\, d\mu(y) + \beta^* \sigma^* \\
\alpha(y) w(y) &\in \kappa(y) \sigma^* - \gamma(y) \mathbf{sign}(\partial_t w(y)) \qquad \mu - a.e. \quad y \in I
\end{aligned}$$

where β^* is the expected value of β.

Theorem (S. 2009, Cont. Mech. Therm.)

Under ergodicity assumptions, the functions u^ε converge to the unique solution u^* almost surely.

Main result in hydrology: Expected pressure

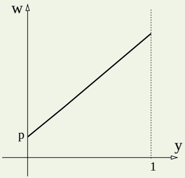
The pressure has bounded gradients, hence $\longrightarrow p^\varepsilon$ without oscillations.
The saturation s^ε, instead, oscillates.

A new quantity: The expected capillary pressure
$w^\varepsilon := a^\varepsilon s^\varepsilon + b^\varepsilon = p_{c,0}(s^\varepsilon)$.

1. case: saturation decreases. Then

$s^\varepsilon = \frac{p^\varepsilon - b^\varepsilon + \gamma^\varepsilon}{a^\varepsilon}$ is oscillatory

The expected pressure is
$w^\varepsilon := a^\varepsilon s^\varepsilon + b^\varepsilon = p^\varepsilon + \gamma^\varepsilon$

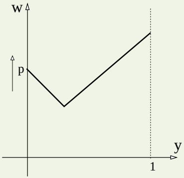


... at places with $\gamma^\varepsilon = y$.

2. case: Variable saturation

Expected pressure

$$w^\varepsilon(x, y, t) := a^\varepsilon s^\varepsilon + b^\varepsilon$$



w encodes the saturation history of the porous medium!

Equations for w

Averaged equations. Conservation law:

$$\partial_t s = \nabla \cdot (K^* \nabla p) \qquad \forall x \in \Omega,$$

where K^* determined by a cell-problem.
The saturation is reconstructed from w via

$$s(x,t) := \int_I \frac{w(x,y,t) - b^*}{a^*}\, dy,$$

where $b^* = \langle b^\varepsilon \rangle$ and $a^* = \langle 1/a^\varepsilon \rangle^{-1}$.
The hysteresis relation holds point-wise,

$$p(x) \in w(x,y) + y\, \mathbf{sign}(\partial_t w(x,y)) \qquad \forall x \in \Omega, y \in [0,1].$$

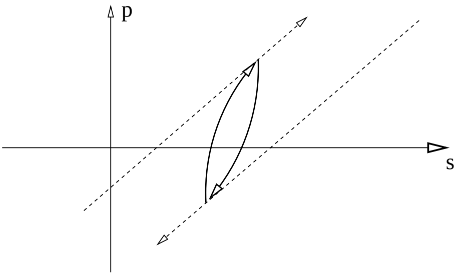
Theorem (S. 2004/07, Eur. J. Appl. Math.)

The equations posess a unique weak solution (s,w,p).
For solutions $(s^\varepsilon, p^\varepsilon)$ of the stochastic ε-problem we have

$$s^\varepsilon \rightharpoonup s, \quad p^\varepsilon \rightharpoonup p \qquad \textit{almost surely.}$$

Scanning curves for the effective equations.

The presence of the history variable w alters the scanning curves.



Hysteresis-cell-problem and operator-cell-problem decouple:

Play-type hysteresis *averages to* **Prandtl-Ishlinskii-hysteresis**

Francu and Krejci (1999): 1-dim. deterministic wave-eq.

Visintin (02-), Alber (09): n-dim. deterministic static

S.-Veneroni (09): n-dim. deterministic wave-eq.

The effective model has scanning curves that are qualitatively as in the experiment.

Desirable for realistic modelling!

Oscillating test-functions

The "correct" description of porous media:

- $\chi_1 : \mathbb{R}^d \to I = [0,1]$ describes the distribution of material, it is chosen stochastically, e.g. constant in unit cubes.
- Heterogeneous media: $\chi_\varepsilon : \Omega \to [0,1], x \mapsto \chi_1(x/\varepsilon)$
- Parameters depend on the material: $a^\varepsilon(x) = a_0(\chi_\varepsilon(x))$ etc.

Method of oscillating test-functions for plasticity. Knowing the homogenized solution (u^*, σ^*, w), we may construct

$$w^\varepsilon(t,x) := w(t,x,\chi^\varepsilon(x)).$$

We expect w^ε to be similar to ϵ^ε.

$$E(t) = \frac{1}{2}\int_\Omega |\partial_t u^\varepsilon - \partial_t u^*|^2 + \frac{1}{2}\int_\Omega \frac{\alpha^\varepsilon}{\kappa^\varepsilon}|\epsilon^\varepsilon - w^\varepsilon|^2 + \frac{1}{2}\int_\Omega \beta^\varepsilon|\sigma^\varepsilon - \sigma^*|^2$$

A direct calculation gives

$$E(t) \le \int_{\Omega_t} \left\{ \left(\int_I \partial_t w(y)\, d\mu(y) - \partial_t w^\varepsilon \right) - (\beta^\varepsilon - \beta^*)\partial_t \sigma^* \right\} (\sigma^\varepsilon - \sigma^*)$$

Stochastic choice of χ_1

To describe stochastic media, one chooses χ_1 stochastically.
Let μ be the distribution of values of χ_1.

Loose definition of ergodicity

The stochastic process is ergodic, if spatial averages yield the expected values (almost surely).

The ergodicity of the medium implies

Definition (Ergodicity property)

Let $g \in L^q(I, d\mu)$ for $q \geq 1$ and let $g^\varepsilon : \Omega \to \mathbb{R}$ be defined as

$$g^\varepsilon(x) = g(\chi^\varepsilon(x)).$$

Then g^ε converges weakly to a constant function,

$$g^\varepsilon(x) \rightharpoonup \langle g \rangle \text{ in } L^q(\Omega) \text{ almost surely.}$$

Two-scale ergodicity

Definition (Two-scale ergodicity property)

We say that the stochastic process and a function $g : \Omega \times I \to \mathbb{R}$ satisfy the *two-scale ergodicity property with* $q \in [1, \infty)$ if the following holds. Consider $g^\varepsilon : \Omega \to \mathbb{R}$ and $\langle g \rangle : \Omega \to \mathbb{R}$,

$$g^\varepsilon(x) = g(x, \chi^\varepsilon(x)), \quad \langle g \rangle (x) = \int_I g(x, y)\, d\mu(y).$$

Then

$$g^\varepsilon \rightharpoonup \langle g \rangle \text{ for } \varepsilon \to 0 \text{ in } L^q(\Omega) \text{ almost surely.}$$

The pair is two-scale ergodic when χ is ergodic and

- g is continuous **or**
- μ has finite support (finite number of materials).

This is the case in the discrete approximation!

Conclusions

- The homogenized system is the one you had guessed in the first place ... once you understood the system well.
- Method of oscillating test-functions is very powerful for rigorous results
- Technical problems are BV-controls and two-scale ergodicity.

Further steps:

- fingering in porous media
- improved existence in porous media hysteresis
- periodic coefficients for plasticity in $\mathbb{R}^n$ ($\longrightarrow$ Marco Veneroni)

Open problem:

- Stochastic coefficients for plasticity in $\mathbb{R}^n$

Thank you!