A free boundary problem in backward diffusion

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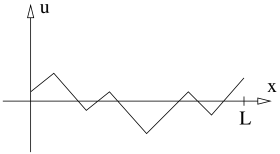
A nonconvex energy functional

$\Omega = (0, L) \subset \mathbb{R}^1$, solution $u : \Omega \to \mathbb{R}$.
Minimize the energy

$$E(u) := \int_\Omega |u|^2 + (1 - |\partial_x u|^2)^2$$

among Lipschitz-functions
$u \in \mathrm{Lip}(\Omega, \mathbb{R})$.



an element of a minimizing sequence

Properties. $E \geq 0$ and $\inf_u E(u) = 0$.
Let u_n be a minimizing sequence. Then:
$u_n \to 0$ in $L^2(\Omega)$ and
$u_n \rightharpoonup 0$ in $W^{1,4}(\Omega)$.

Two problems

- $u = 0$ is not a minimizer
- loss of information

Fundamental idea: identify functions $f : \Omega \to \mathbb{R}$ with the family of Radon-measures $\nu_x \in \mathcal{M}(\mathbb{R})$,

$$\nu_x = \delta_{f(x)} \qquad \forall x \in \Omega.$$

- define $\nu_x^n \in \mathcal{M}(\mathbb{R})$ as

$$\nu_x^n = \delta_{\partial_x u_n(x)}$$

- The sequence ν^n is bounded (by 1) in

$$X = L^\infty(\Omega; \mathcal{M}(\mathbb{R}))$$

- X is the dual of $L^1(\Omega; C_0(\mathbb{R}))$. Hence, for a subsequence,

$$\nu^n \stackrel{*}{\rightharpoonup} \nu$$

  for some $\nu \in X$.

In one formula: For $f \in C_0(\mathbb{R})$ and $\varphi \in C_c(\Omega)$

$$\int_\Omega f(\partial_x u_n(x))\varphi(x)\,dx \to \int_\Omega \int_\mathbb{R} f(\lambda)\,d\nu_x(\lambda)\varphi(x)\,dx.$$

In our example we find

$$\nu_x = \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_{+1}$$

the Young measure is the limiting probability distribution of values

Idea of proof: The formula implies

$$f(\partial_x u_n(.)) \rightharpoonup \int_{\mathbb{R}} f(\lambda)\, d\nu_x(\lambda).$$

Use $f(p) = (1 - |p|^2)^2$. Then

$$0 \leftarrow E(u_n) \geq \int_\Omega (1 - |\partial_x u_n|^2)^2,$$

hence

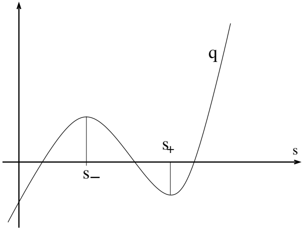
$$0 \leftharpoondown f(\partial_x u_n) \rightharpoonup \int_\Omega \int_{\mathbb{R}} f(\lambda)\, d\nu_x(\lambda)\, dx.$$

The only zeroes of f are $+1$ and -1.
For the probabilities: Use $f(p) = p$.

Our problem

$$\partial_t u = \partial_x[q(\partial_x u)]$$

on $\Omega_T = \Omega \times (0,T)$ for $\Omega = (0,L) \subset \mathbb{R}^1$.



We may write

$$\partial_t u = q'(\partial_x u)\partial_x^2 u$$

Backward diffusion in regions with $\partial_x u(x) \in (s_-, s_+)$.

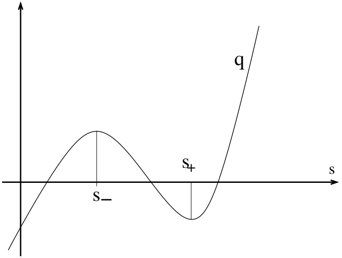
[1] M. Slemrod, Dynamics of Measure Valued Solutions to a Backward-Forward Heat Equation. J. Dyn. Differ. Eqns. 3, 1–28 (1991).

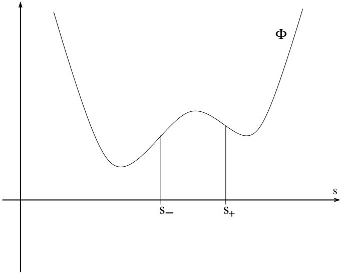
[2] S. Demoulini, Young measure solutions for a nonlinear parabolic equation of forward-backward type. SIAM J. Math. Anal. 27, 376-403 (1996).

[3] D. Horstmann and B. Schweizer, A free boundary characterization for forward-backward diffusion. Advances in Differential Equations, (2008).

We can construct the primitive for q, the function $\Phi : \mathbb{R} \to \mathbb{R}$ with

$$\Phi'(s) = \partial_s \Phi(s) \overset{!}{=} q(s).$$



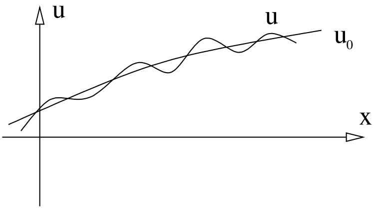


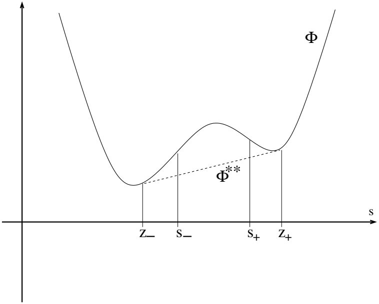
Energy decay: $E(t) := \int_\Omega \Phi(\partial_x u(t,x))\, dx$ satisfies

$$\partial_t E(t) = \int_\Omega \Phi'(\partial_x u)\partial_t \partial_x u = -\int_\Omega \partial_x[q(\partial_x u)]\partial_t u = -\int_\Omega |\partial_t u|^2$$

Our problem:

$$q \text{ not monotone } \longleftrightarrow \Phi \text{ not convex}$$





Weak solution concept for $\partial_t u = \partial_x[q(\partial_x u)]$. $q^* = \partial_s \Phi^{**}(z_-)$

Definition: Young-measure solution (simplified)

(u, ν) is a *Young measure solution* if
$u \in H^1(\Omega_T)$ and $\nu : \Omega_T \to \mathcal{M}(\mathbb{R})$satisfy

$$\partial_t u = \nabla \cdot \bar{q} \quad \text{in the sense of distributions with} \tag{1}$$

$$\bar{q}(t,x) = \langle \nu_{t,x}, q \rangle\,, \tag{2}$$

$$\nabla u(t,x) = \langle \nu_{t,x}, \mathrm{id}_{\mathbb{R}} \rangle \quad \text{for a.e. } (t,x) \in \Omega_T. \tag{3}$$

Young measure solutions (u, ν) satisfy

$$\partial_t u = \nabla \cdot \bar{q} \quad \text{in the sense of distributions with}$$
$$\bar{q}(t,x) = \langle \nu_{t,x}, q \rangle ,$$
$$\nabla u(t,x) = \langle \nu_{t,x}, \mathrm{id}_{\mathbb{R}} \rangle \quad \text{for a.e. } (t,x) \in \Omega_T .$$

An observation: Frozen solutions.

Let the initial values satisfy $\partial_x u_0(x) \in (z_-, z_+)$ for all x.

$$u(t,x) := u_0(x) \quad \forall t \in [0,T],$$
$$\nu_{t,x} := \sigma(x)\delta_{z_-} + (1-\sigma(x))\delta_{z_+}.$$

where σ is determined such that $\sigma(x)z_- + (1-\sigma(x))z_+ = \partial_x u_0(x)$. Then (u, ν) is a Young measure solution.

- The definition of ν implies $\langle \nu, \mathrm{id} \rangle = \partial_x u$, hence (3)
- We set $\bar{q}(t,x) := \langle \nu_{t,x}, q \rangle = \sigma(x)q(z_-) + (1-\sigma(x))q(z_+) = q^*$ according to (2).
- Then $\partial_t u = 0 = \partial_x \bar{q}$, hence (1).

Slemrod An SP-solution (u, ν) is constructed from solutions u_ε of

$$\partial_t u^\varepsilon = \partial_x[q(\partial_x u^\varepsilon)] - \varepsilon^2 \partial_x^4 u^\varepsilon.$$

$u_\varepsilon \to u$ almost everywhere and $\partial_x u_\varepsilon$ generates $\nu_{t,x}$.

Demoulini An EM-solution (u, ν) is constructed with a time-discretization. Given $u^{(t)}$, find $u = u^{(t+\Delta t)}$ through

$$\int_\Omega \Phi(\partial_x u) + \frac{1}{2}\frac{1}{\Delta t}\int_\Omega |u - u^{(t)}|^2 \to \min.$$

Weak limits are again Young-measure solutions. Furhermore, they satisfy:

$$\nu_{x,t} \text{ is of the form } \begin{cases} \delta_{\partial_x u(x,t)} \text{ or} \\ \sigma(x,t)\delta_{z_-} + (1-\sigma(x,t))\delta_{z_+} \end{cases}$$

independence property

$$\langle \nu_{t,x}, q^{**} \rangle = \langle \nu_{t,x}, q \rangle$$
$$\langle \nu_{t,x}, \mathrm{id}_{\mathbb{R}} \cdot q^{**} \rangle = \langle \nu_{t,x}, \mathrm{id}_{\mathbb{R}} \rangle \cdot \langle \nu_{t,x}, q^{**} \rangle$$

The independence property makes the Young measure solution unique. Let $(\bar{u}, \bar{\nu})$ with $\bar{q} = \langle \bar{\nu}, q \rangle$, and $(\tilde{u}, \tilde{\nu})$ with $\tilde{q} = \langle \tilde{\nu}, q \rangle$ be solutions. We multiply $\partial_t(\bar{u} - \tilde{u}) = \nabla \cdot (\bar{q} - \tilde{q})$ with $\bar{u} - \tilde{u}$. The right hand side is

$$\int_{\Omega_T} (\bar{q} - \tilde{q}) \cdot \nabla(\bar{u} - \tilde{u})$$
$$= \int_{\Omega_T} \int_{\mathbb{R}^2} q(\lambda_1)\lambda_2 \, d\bar{\nu}_{t,x}(\lambda_1) d\bar{\nu}_{t,x}(\lambda_2) - \int_{\Omega_T} \int_{\mathbb{R}^2} q(\lambda_1)\lambda_2 \, d\bar{\nu}_{t,x}(\lambda_1) d\tilde{\nu}_{t,x}(\lambda_2)$$
$$- \int_{\Omega_T} \int_{\mathbb{R}^2} q(\lambda_2)\lambda_1 \, d\bar{\nu}_{t,x}(\lambda_1) d\tilde{\nu}_{t,x}(\lambda_2) + \int_{\Omega_T} \int_{\mathbb{R}^2} q(\lambda_1)\lambda_2 \, d\tilde{\nu}_{t,x}(\lambda_1) d\tilde{\nu}_{t,x}(\lambda_2)$$
$$= \int_{\Omega_T} \int_{\mathbb{R}^2} [q^{**}(\lambda_1) - q^{**}(\lambda_2)] \cdot (\lambda_1 - \lambda_2) \, d\bar{\nu}_{t,x}(\lambda_1) d\tilde{\nu}_{t,x}(\lambda_2) \geq 0.$$

Collecting facts ...

- There exists a Young-measure solution, constructed with a viscosity approach (Slemrod). Comparison principle. No uniqueness.
- There exists a Young-measure solution which, additionally, satisfies the independence property (Demoulini). This solution is unique.

The solutions can be different

Let u_0 satisfy $\partial_x u_0(x) \in (z_-, s_-)$ for all x. Then

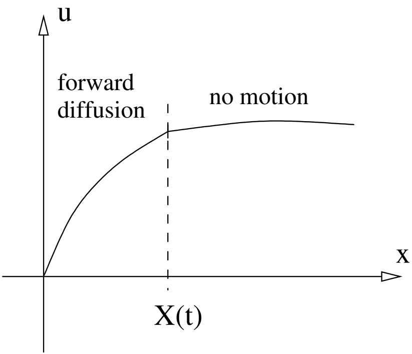
- Forward regime $\longrightarrow$ classical solution exists $\longrightarrow$ Slemrod concept recovers classical solution.
- Independence does not allow $\partial_x u \in (z_-, z_+)$. Result:

$$u(., t) = u_0.$$

Stabilization [Demoulini]. Given u_0, there exists $U \in H^1(\Omega)$ such that

$$u(x,t) \to U(x) \text{ for } t \to \infty.$$

For EM-solutions we expect:



The free boundary problem

$$X(0) = x_0, \partial_x u_0(x_0) = z_+$$
$$\partial_t u(t,\cdot) = 0 \text{ for } x > X(t)$$
$$\partial_t u(t,\cdot) = \partial_x[q(\partial_x u)] \text{ for } x < X(t)$$
$$\partial_x u(t, X(t) - 0) = z_+$$

Theorem (Characterization of the EM-solution)

$u_0 \in C^2(\bar\Omega)$ with $u_0(0) = u_0(L) = 0$ and

$$\begin{aligned} \partial_x u_0 &\geq z_+ \text{ in } [0, x_1), \\ \partial_x u_0 &\in (z_-, z_+) \text{ in } (x_1, x_2), \\ \partial_x u_0 &\leq z_- \text{ in } (x_2, L]. \end{aligned}$$

Then the EM-solution is given by the unique solution of (P).

(P): Find $u \in C(\bar\Omega \times [0,T], \mathbb{R}) \cap L^2((0,T), H_0^1(\Omega, \mathbb{R}))$, and $X_1, X_2 \in C([0,T],[0,L]) \cap W^{1,1}((0,T),\mathbb{R})$, X_1 increasing, X_2 decreasing,

$$\begin{aligned} &X_1(0) = x_1,\ X_2(0) = x_2, \\ &\partial_t u(t,\cdot) = 0 \text{ in } (X_1(t), X_2(t)), \\ &\partial_t u(t,\cdot) = \partial_x[q(\partial_x u)] \text{ in } (0,L) \setminus (X_1(t), X_2(t)), \\ &\partial_x u(t, X_1(t) - 0) = z_+, \partial_x u(t, X_2(t) + 0) = z_-. \end{aligned}$$

Ideas for the proof.

1. We set $v := \partial_x u - z$ and differentiate the equation to find

$$\partial_t v = \partial_x [q'(v+z)\partial_x v] =: \partial_x [a \partial_x v]$$

   The direct boundary condition in $X(t)$ is $v(X(t)) = 0$.

2. The boundary condition $u(t, X(t)) = u_0(X(t))$ gives

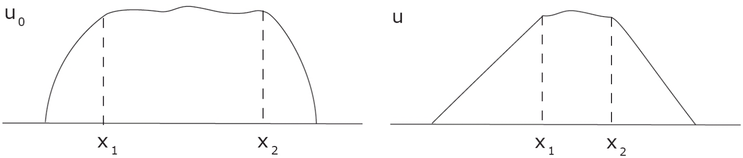
$$\partial_x u \, \partial_t X + \partial_t u = \partial_x u_0 \, \partial_t X$$

   or, subtracting z, and using $v_0 = \partial_x u_0 - z$

$$v \partial_t X + a \partial_x v = v_0 \partial_t X$$

3. v is positive for $x < X(t)$ and v_0 is negative for $x > X(t)$. Nevertheless, we can make the equation explicit for $\partial_t X$.

Easy direction: Given a solution to the free boundary problem $\longrightarrow$ this is the EM-solution.



Theorem (Long-time behavior)

Let initial values $u_0 \in C^2(\bar{\Omega})$ *be as before. Then,*

1. *If* $u_0(\xi) \geq \min\{z_+\xi, z_-(\xi - L)\}$ *for all* $\xi \in (0, L)$*, then the EM-solution* u *exists for all times and*

$$u(x,t) \to \min\{u_0(x), z_+x, z_-(x-L)\} \text{ for } t \to \infty.$$

2. *If* $u_0(\xi) < \min\{z_+\xi, z_-(\xi - L)\}$ *for one* $\xi \in (0, L)$*, then*

$$u(x,t) \to \min\{z_+x, z_-(x-L)\} \text{ for } t \to \infty.$$

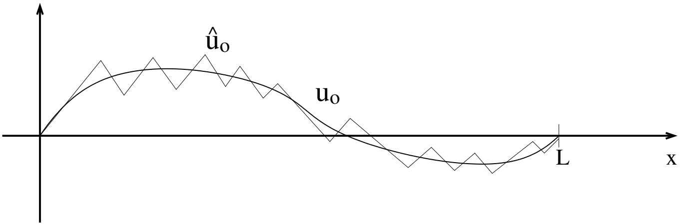
Theorem (Discontinuous dependence of the SP-solution)

Let $u_0 \in C^1(\bar{\Omega})$ *satisfy* $z_- < \partial_x u_0 < z_+$*, with EM-solution* $u_{EM}(x,t) = u_0(x)$.
There exists a sequence of perturbations $u_0^\delta \in C^1(\bar{\Omega})$ *of the initial values,*

$$\|u_0^\delta - u_0\|_{L^\infty} \to 0 \text{ for } \delta \to 0,$$

such that every corresponding SP-solution $u_{SP}^\delta : \Omega \times [0,T] \to \mathbb{R}$ *satisfies*

$$u_{SP}^\delta(t,\cdot) \to u_{EM} \text{ in } L^\infty(\Omega) \text{ for } \delta \to 0.$$



Thank you!