Time harmonic Maxwell's equations in periodic waveguides

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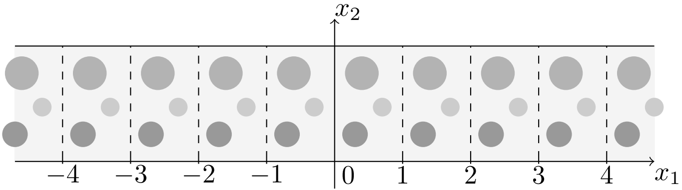
Maxwell's equations in a wave-guide

Given: Frequency $\omega > 0$, permeability $\mu = \mu(x)$, permittivity $\varepsilon = \varepsilon(x)$, a right hand side (f_h, f_e)

Goal: Solve **Maxwell's equations**

$$\begin{aligned} \operatorname{curl} E &= i\omega\mu\, H + f_h \\ \operatorname{curl} H &= -i\omega\varepsilon\, E + f_e \end{aligned} \tag{1}$$

in a **waveguide geometry:** $\Omega = \mathbb{R} \times S \subset \mathbb{R}^3$
with $S \subset \mathbb{R}^2$ a bounded Lipschitz domain



The simplest related equation

An ordinary differential equation on $\mathbb{R}$

Given $f : \mathbb{R} \to \mathbb{C}$ with support in $[-R, R]$ and $\omega > 0$,
find $u : \mathbb{R} \to \mathbb{C}$ solving

$$\partial_x^2 u + \omega^2 u = f$$

Solution: For some $a_1, a_2, b_1, b_2 \in \mathbb{C}$:
For $x > R$, the solution is $u(x) = a_1 e^{i\omega x} + a_2 e^{-i\omega x}$
For $x < -R$, the solution is $u(x) = b_1 e^{i\omega x} + b_2 e^{-i\omega x}$

Radiation condition: One might want, e.g., $a_2 = b_1 = 0$

Robin conditions in $x = R$ and $x = -R$ provide the solution u

Note: In general, $u \notin H^1(\mathbb{R}, \mathbb{C})$

Reformulation of Maxwell's equations

Goal: For $\omega > 0$, $\mu = \mu(x)$, $\varepsilon = \varepsilon(x)$, $(f_h, f_e) = (f_h, f_e)(x)$, solve

$$\operatorname{curl} E = i\omega\mu\, H + f_h$$
$$\operatorname{curl} H = -i\omega\varepsilon\, E + f_e$$

with the boundary conditions for perfect conductors: $E \times \nu = 0$ on $\partial\Omega$

Periodicity, positivity: $\varepsilon, \mu \in L^\infty(\Omega)$ strictly positive, 2π-periodic in x_1

Decay property of the right hand side: $\int_\Omega (1 + x_1^2)^2 \, |f(x)|^2 dx < \infty$

Weak formulation with $u := H$ as the only unknown

$$\int_\Omega \left\{ \frac{1}{\varepsilon} \operatorname{curl} u \cdot \operatorname{curl} \bar{\phi} - \omega^2 \mu\, u\, \bar{\phi} \right\} = \int_\Omega \left\{ \frac{1}{\varepsilon} f_e \operatorname{curl} \bar{\phi} - i\omega f_h\, \bar{\phi} \right\} \qquad (2)$$

for every $\phi \in H^1(\Omega, \mathbb{C}^3)$ with bounded support

This already encodes the boundary conditions

Useful function space:

$$u \in H(\operatorname{curl}, \Omega) := \left\{ u \in L^2(\Omega, \mathbb{C}^3) \middle| \operatorname{curl} u \in L^2(\Omega, \mathbb{C}^3) \right\}$$

Floquet-Bloch transformation

Periodicity cell: $W := (0, 2\pi) \times S$ for $\Omega = \mathbb{R} \times S$

Quasimoments: $\alpha \in I := [-1/2, 1/2]$

Floquet-Bloch transform

$$\mathcal{F}_{\mathrm{FB}} : L^2(\Omega) \to L^2\big(W \times I\big), \qquad u = u(x) \mapsto \hat{u} = \hat{u}(x, \alpha)$$

For smooth functions u with compact support, $x = (x_1, \tilde{x})$:

$$\hat{u}((x_1, \tilde{x}), \alpha) := \sum_{\ell \in \mathbb{Z}} u(x_1 + 2\pi\ell, \tilde{x})\, e^{-i\ell 2\pi\alpha}$$

Inverse: For arbitrary $y \in \Omega$, reconstruct with

$$u(y) = \int_I \hat{u}(y, \alpha)\, d\alpha$$

With these formulas also: $\mathcal{F}_{\mathrm{FB}}$ is an isomorphism with bounded inverse

$$\mathcal{F}_{\mathrm{FB}}^{-1} : L^2\big(I, H^1_\alpha(W)\big) \to H^1(\Omega)$$

Floquet-Bloch transformed equation

$W := (0, 2\pi) \times S\,, \qquad \alpha \in I := [-1/2, 1/2]$

Function space: $X := H_{\mathrm{per}}(\mathrm{curl}\,, W)$ with scalar product

$$\langle u, \phi\rangle_X := \left\langle \frac{1}{\varepsilon}\mathrm{curl}\, u, \mathrm{curl}\, \phi \right\rangle_{L^2(W)} + \langle \mu u, \phi\rangle_{L^2(W)}$$

The Maxwell operator

For every $\alpha \in I$, a linear operator $L_\alpha : X \to X$ is defined by

$$\langle L_\alpha v, \varphi\rangle_X := \int_W \frac{1}{\varepsilon}\mathrm{curl}\,(v e^{i\alpha x_1}) \cdot \mathrm{curl}\,\overline{(\varphi e^{i\alpha x_1})} - \omega^2 \mu\, v \cdot \bar{\varphi}$$

The right hand side (f_h, f_e) is represented with a family $y_\alpha \in X$

Equivalent formulation

Maxwell is solved with $u \in H^1(\Omega)$ when we show: For almost every $\alpha \in I$, there is $v(\cdot, \alpha) \in X$ solving

$$L_\alpha v(\cdot, \alpha) = y_\alpha \tag{3}$$

and there holds $v \in L^2(I, X)$

Critical α-values

Trivial case: When L_α^{-1} exists for all α, then $v(\cdot,\alpha) = L_\alpha^{-1}(y_\alpha)$

Critical α-values

For $\alpha \in I$ let Y^α be the space of α-quasiperiodic solutions to the homogeneous problem. Critical values:

$$\mathcal{A} \;:=\; \{\alpha \in [-1/2, 1/2] \,|\, Y^\alpha \neq \{0\}\}$$

$\longrightarrow$ Y^α consists of propagating modes

Energy transport is encoded with an hermitean form:

$$Q(u,\phi) := i \int_W \frac{1}{\varepsilon} \left[(\operatorname{curl} u \times \bar{\phi}) - (\operatorname{curl} \bar{\phi} \times u)\right] \cdot e_1$$

u a propagating mode

u transports energy to the right $\iff$ $Q(u,u) > 0$
u transports energy to the left $\iff$ $Q(u,u) < 0$

Assumption

For every $0 \neq \phi \in Y^\alpha$, the map $Q(\cdot,\phi) : Y^\alpha \to \mathbb{C}$ is non-trivial

Functional analysis

Definition (Regular C^1-family)

$(L_\alpha)_\alpha$ is a regular C^1-family when:

1. L_α is a self-adjoint Fredholm operator with index 0 (for every α)
2. The operators depend differentiable on α
3. The derivatives $\partial_\alpha L_\alpha$ are invertible on the kernel for every α

Theorem (Functional analysis, Kirsch et al.)

Let $(L_\alpha)_\alpha$ be a regular C^1-family of operators
Let $\alpha \mapsto y_\alpha \in L_\alpha(X)$ map into the image and be Lipschitz
Then $v(\cdot, \alpha) = L_\alpha^{-1}(y_\alpha)$ is uniformly bounded

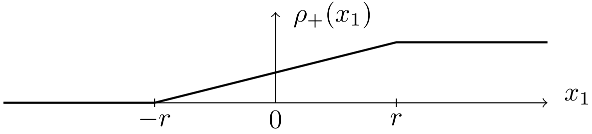
Proof with implicit function theorem

Information regarding Maxwell: $(L_\alpha)_\alpha$ is a regular C^1-family
Relevant step: Fredholm property of L_α
Invertibility of $\partial_\alpha L_\alpha$ on the kernel follows from Q-assumption
(Loosely speaking: Q is the derivative $\partial_\alpha L_\alpha$)

Problem: In general, $y_\alpha \notin L_\alpha(X)$

Indeed: When $y_\alpha \in L_\alpha(X)$ for all α, we find $u \in L^2(\Omega)$!

Cut-off functions: ρ_+ with limits 1 and 0 and $\rho_- := 1 - \rho_+$



Let $(\phi_\ell)_\ell$ be the quasiperiodic homogeneous solutions
For every ℓ, let ρ_ℓ be one of the two cut-off functions

Definition (Propagating part and radiation condition)

(i) Propagating part. For complex coefficients $(a_\ell)_{1\le\ell\le L}$,

$$u^{\mathrm{prop}} := \sum_{\ell=1} a_\ell \, \rho_\ell \, \phi_\ell$$

is the propagating wave function corresponding to $a \in \mathbb{C}^L$

(ii) Radiation condition. A solution $u \in H_{\mathrm{loc}}(\mathrm{curl}\,, \Omega)$ of (2) satisfies the radiation condition, when there exists $a \in \mathbb{C}^L$ such that

$$u^{\mathrm{rad}} := u - u^{\mathrm{prop}} \in H(\mathrm{curl}\,, \Omega)$$

Existence result

Theorem (Existence and uniqueness of solutions to the radiation problem)

Let S, ω, ε, μ, f_e *and* f_h *be given, let the* Q*-assumption be satisfied. Then (2) has a unique solution* $u \in H_{\mathrm{loc}}(\mathrm{curl}\,,\Omega)$ *satisfying the radiation condition. With* $C = C(S, \varepsilon, \mu, \omega, \rho_\pm)$ *holds*

$$\|u^{\mathrm{rad}}\|_{H(\mathrm{curl}\,,\Omega)} + \|u^{\mathrm{prop}}|_W\|_{H(\mathrm{curl}\,,W)} \leq C\ \left(\|f_e\|_{L^2_*(\Omega)} + \|f_h\|_{L^2_*(\Omega)}\right)$$

The coefficients $(a_\ell)_\ell$ *are given by*

$$a_\ell \ = \ \frac{2\pi i}{|Q(\phi_\ell, \phi_\ell)|}\left(\langle \varepsilon^{-1} f_e, \mathrm{curl}\,\phi_\ell\rangle_{L^2(\Omega)} - \langle i\omega f_h, \phi_\ell\rangle_{L^2(\Omega)}\right)$$

Sketch of proof

Step 1: Under some orthogonality condition holds $y_\alpha \in L_\alpha(X)$ for all α
Functional analysis theorem yields the solution in $u \in H^1(\Omega)$

Step 2: General case. Pre-factors $(a_\ell)_\ell \in \mathbb{C}^L$ yield u^{prop}
$u - u^{\mathrm{prop}}$ satisfies a different problem. Show that $a_\ell \in \mathbb{C}$ can be chosen such that the problem for $u - u^{\mathrm{prop}}$ satisfies the orthogonality condition.

Fredholm property

Here: $\alpha \in I$ is fixed, any dependence on α is suppressed

$W = (0, 2\pi) \times S$, $\quad \varepsilon, \mu \in L^\infty(\Omega)$ real valued and positive

Function space: $H_\alpha(\operatorname{curl}, W)$ with scalar product

$$\langle u, \varphi \rangle_{H(\operatorname{curl}, W)} := \int_W \left\{ \frac{1}{\varepsilon} \operatorname{curl} u \cdot \operatorname{curl} \bar{\varphi} + \mu\, u \cdot \bar{\varphi} \right\}$$

Operator: $L : H_\alpha(\operatorname{curl}, W) \to H_\alpha(\operatorname{curl}, W)$ defined by

$$\langle Lu, \varphi \rangle_{H(\operatorname{curl}, W)} = \int_W \left\{ \frac{1}{\varepsilon} \operatorname{curl} u \cdot \operatorname{curl} \bar{\varphi} - \omega^2 \mu\, u \cdot \bar{\varphi} \right\}$$

Helmholtz decomposition: $H_\alpha(\operatorname{curl}, W) = D \oplus G$:

$$D := \left\{ u \in H_\alpha(\operatorname{curl}, W) \,\middle|\, \int_W \mu\, u \cdot \nabla \psi = 0 \text{ for all } \psi \in H^1_\alpha(W) \right\}$$

$$G := \left\{ v \in H_\alpha(\operatorname{curl}, W) \,\middle|\, \exists \psi \in H^1_\alpha(W) : \ v = \nabla \psi \right\}$$

$\longrightarrow H(\operatorname{curl}, W)$-orthogonal complements

Fredholm property

Lemma (Fredholm property)

The operator L is a self-adjoint Fredholm operator with index 0.

Consider $v \in G$ and $Lv \in X$ and $u \in D$:

$$\langle Lv, u \rangle_{H(\mathrm{curl}\,, W)} = -\omega^2 \langle \mu v, u \rangle_{L^2(W)} = -\omega^2 \langle v, \mu u \rangle_{L^2(W)} = 0$$

This provides $L|_G : G \to G$. Similarly: $L|_D : D \to D$.
Hence, on $H_\alpha(\mathrm{curl}\,, W) = D \oplus G$:

$$L = \begin{pmatrix} L|_D & 0 \\ 0 & L|_G \end{pmatrix}$$

$L|_D : D \to D$ is a Fredholm operator with index 0: One shows that $K := L - \mathrm{id}$ is a compact operator $D \to D$.

On G, the operator L is nothing but multiplication with $-\omega^2$, hence a Fredholm operator with index 0.

$Y = B$

Two function spaces of modes

The span of quasiperiodic solutions

$$Y \ := \ \bigoplus_{j=1}^{J} Y_j \subset H(\mathrm{curl}\,, W)\,, \quad \text{identified with} \quad Y \ \subset \ H_{\mathrm{loc}}(\mathrm{curl}\,, \Omega)$$

The space of bounded solutions, $\|U\|_{sL} := \sup_{r \in 2\pi\mathbb{Z}} \|U|_{W_r}\|_{L^2(W_r)}$:

$$B \ := \ \{U \in H_{\mathrm{loc}}(\mathrm{curl}\,, \Omega) \,|\, U \text{ solves (2) for } f_e = f_h = 0\,,\ \|U\|_{sL} < \infty\}$$

Theorem (Characterization of bounded homogeneous solutions)

When the Q-assumption is satisfied, then

$$Y \ = \ B$$

Proof of $Y = B$.

We consider $U \in B$ and want to show $U \in Y$

Let $f = f_h$ with compact support be arbitrary

For f, let the Maxwell solution be $u = u^{\mathrm{prop}} + u^{\mathrm{rad}}$ with $(a_\ell)_{1 \le \ell \le L}$

With cut-off function ϑ_R, use $U\vartheta_R$ as a test-function

$$\int_\Omega \left\{ \frac{1}{\varepsilon} \mathrm{curl}\, u \cdot \mathrm{curl}\, (\bar{U}\vartheta_R) - \omega^2 \mu\, u \cdot \bar{U}\vartheta_R \right\} = -i\omega \int_\Omega f \cdot \bar{U}$$

Evaluate the left hand side using that U is a homogeneous solution:
With c_ℓ depending on U and ϕ_ℓ, but not on f, we find

$$\sum_{\ell=1}^{L} c_\ell\, a_\ell = -i\omega \, \langle f, U \rangle_{L^2(\Omega)}$$

We now recall: a_ℓ is a linear combination of $\langle f, \phi_k \rangle_{L^2(\Omega)}$.

Result, since f was arbitrary: U is a linear combination of the ϕ_k.

Locally perturbed media

Theorem (Fredholm alternative for perturbed media)

Let $\mu_{\rm per}, \varepsilon_{\rm per} \in L^\infty(\Omega)$ be periodic functions with positive lower bounds

Let $\mu, \varepsilon \in L^\infty(\Omega)$ be given as compact perturbations of $\mu_{\rm per}, \varepsilon_{\rm per}$
We assume positive lower bounds also for μ, ε

Let the Q-assumption be satisfied for $\mu_{\rm per}, \varepsilon_{\rm per}$

Let $u = 0$ be the only solution to the homogeneous perturbed system

Then there exists a unique radiating solution for every (f_e, f_h)

Hint on the proof: With operators $D : X \to Y$ and $\xi, Q : Y \to Y$

$$D := \begin{pmatrix} 0 & \mathrm{curl} \\ -\mathrm{curl} & 0 \end{pmatrix}, \quad \xi := \begin{pmatrix} \varepsilon_{\rm per} & 0 \\ 0 & \mu_{\rm per} \end{pmatrix}, \quad Q := \begin{pmatrix} q_\varepsilon & 0 \\ 0 & q_\mu \end{pmatrix}$$

Maxwell's equations take the form

$$(D + i\omega\,\xi)u = i\omega\,Qu + f$$

Show Fredholm property for compactly supported q_ε, q_μ

$\longrightarrow$ Helmholtz decompositions

Conclusions

The radiation problem for time harmonic Maxwell's equations in wave-guides can be solved

- Method: Functional analysis (implicit function theorem)
- Underlying operator is Fredholm (for fixed quasi-moment α)

Further results:

- Compactly perturbed media
- $Y = B$

Thank you!

Recent articles:

Kirsch A and Schweizer B (2024), "Time harmonic Maxwell's equations in periodic waveguides" (submitted)

Kirsch A and Schweizer B (2024), "Periodic wave-guides revisited: Radiation conditions, limiting absorption principles, and the space of boundes solutions", Mathematical Methods in the Applied Sciences