Representation Formulas for Wave Equations with Profile Functions

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Overview

- Part 1: Three wave equations, long-time-dispersion
  - Wave equation with constant coefficient
  - Wave equation in periodic media
  - Wave equation on a lattice
- Part 2: Formulas for profiles

Wave equation with constant coefficient

Wave equation

$$\partial_t^2 u(x,t) = c^2 \Delta u(x,t)$$

Coefficient $a = c^2$ independent of x

Planar-wave ansatz: $u(x,t) = e^{ik\cdot x} e^{-i\omega t}$

u is a solution if and only if $\omega^2 = c^2 |k|^2$

Result: The dispersion relation is $\omega(k) = \pm c|k|$

A solution:

$$u(x,t) = \int_{\mathbb{R}^d} u_0(k) e^{ik\cdot x} e^{-ic|k|t}\, dk$$

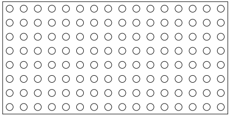
Satisfy initial conditions by using $+$ and $-$

One dimensional equation: An exact solution is given by shifts,

$$u(x,t) = \tfrac{1}{2} u_0(x-ct) + \tfrac{1}{2} u_0(x+ct)$$

Compare Fourier representation: Solutions depend on $x+ct$ and $x-ct$

Wave equation in periodic media



Let $a : \mathbb{R}^d \to (\delta, \infty)$ be 1-periodic
Set $a_\varepsilon(x) := a(x/\varepsilon)$

Homogenization problem

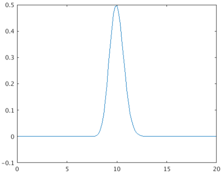
$$\partial_t^2 u^\varepsilon = \nabla \cdot (a_\varepsilon(x) \nabla u^\varepsilon(x))$$

Classical homogenization: $u^\varepsilon \approx u$, where u solves the

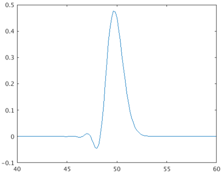
Homogenized equation

$$\partial_t^2 u(x,t) = \nabla \cdot a_* \nabla u(x,t)$$

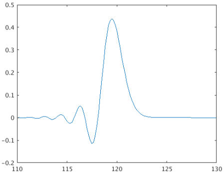
Observation in dimension 1 with periodicity $\varepsilon = 1/6$, the solution of homogenized equation is a shift, $u(x,t) = u_0(x-t)$



Time $t = 10$



Time $t = 50$



Time $t = 120$

$\longrightarrow$ We observe dispersion for large times!

Result on weak dispersion

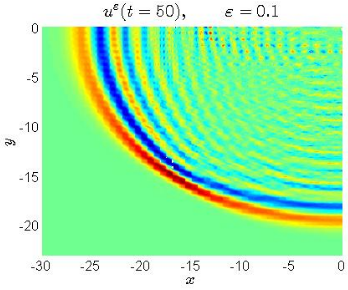
Theorem (T.Dohnal, A.Lamacz, B.S., 2014 and 2015)

There exist $A, E \in \mathbb{R}^{d\times d}$ and $F \in \mathbb{R}^{d\times d\times d\times d}$ such that:
Solutions of the well posed weakly dispersive effective equation

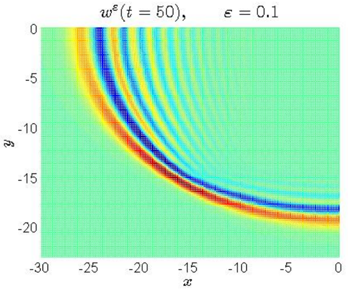
$$\partial_t^2 w^\varepsilon = AD^2 w^\varepsilon + \varepsilon^2 ED^2\partial_t^2 w^\varepsilon - \varepsilon^2 FD^4 w^\varepsilon$$

satisfy the approximation estimate

$$\sup_{t\in[0,T\varepsilon^{-2}]} \|u^\varepsilon(\cdot,t) - w^\varepsilon(\cdot,t)\|_{L^2(\mathbb{R}^d)+L^\infty(\mathbb{R}^d)} \le C_0\varepsilon$$



u^ε: solution for coefficient a_ε



w^ε: solution of weakly dispersive equation

Bloch-wave analysis

The proof is based on an explicit formula for solutions:

Proposition (Bloch-wave approximation of u^ε)

$$v^\varepsilon(x,t) := \frac{1}{2\sqrt{2\pi}^n} \sum_{\pm} \int_K F_0(k) e^{\mathrm{i}k\cdot x} \exp\left(\pm \mathrm{i}t\sqrt{\sum A_{lm}k_l k_m}\right)$$

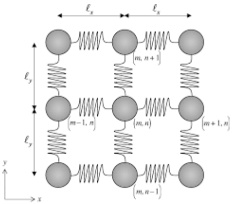
$$\times \exp\left(\pm \frac{\mathrm{i}\varepsilon^2}{2} t \frac{\sum C_{lmnq}k_l k_m k_n k_q}{\sqrt{\sum A_{lm}k_l k_m}}\right) dk$$

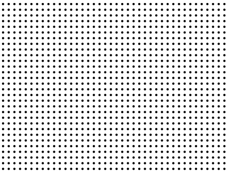
$$\sup_{t\in[0,T\varepsilon^{-2}]} \|u^\varepsilon(\cdot,t) - v^\varepsilon(\cdot,t)\|_{(L^2+L^\infty)(\mathbb{R}^n)} \le C_0\varepsilon$$

- Taylor expansion of the smallest Bloch eigenvalue $\mu_0(k)$
- Replacement trick to re-write the equation
- Estimate for $u^\varepsilon - v^\varepsilon$ from approximations in the explicit formula
- Estimate for $v^\varepsilon - w^\varepsilon$ by energy methods

A.Benoit and A.Gloria. Long-time homogenization and asymptotic ballistic transport of classical waves, 2017

Waves in a lattice





Periodicity of the lattice: $\varepsilon > 0$
Dimension $d \geq 1$, lattice points $\gamma \in \varepsilon\mathbb{Z}^d$
Displacement at time $t \in [0, \infty)$ is $u^\varepsilon(\gamma, t)$.

Evolution of displacements

Wave equation in lattice:

$$\partial_t^2 u^\varepsilon(\gamma, t) = \frac{1}{\varepsilon^2} \sum_{j \in \mathbb{Z}^d} a_j u^\varepsilon(\gamma + \varepsilon j, t)$$

Initial conditions: $u^\varepsilon(\gamma, 0) = u_0(\gamma)$ and $\partial_t u^\varepsilon(\gamma, 0) = u_1(\gamma)$.

Example: Dimension $d = 1$, $a_1 = a_{-1} = 1$, $a_0 = -2$, and $a_j = 0$ else.

Explicit solutions for lattice waves in Fourier space

The equation for the **Fourier transform**

$$\hat{u}^\varepsilon(k,t) = \varepsilon^d \sum_{\gamma \in (\varepsilon\mathbb{Z})^d} e^{-ik\cdot\gamma} u^\varepsilon(\gamma,t)$$

is easily calculated: For every $k \in \mathbb{R}^d$ holds

$$\varepsilon^2 \partial_t^2 \hat{u}^\varepsilon(k,t) = -\omega(\varepsilon k)^2\, \hat{u}^\varepsilon(k,t) \quad \text{with} \quad \omega(k)^2 := -\sum_{j\in\mathbb{Z}^d} a_j e^{ik\cdot j}$$

The **dispersion relation** in the example is $\frac{\omega(\varepsilon k)}{\varepsilon} = \sqrt{k^2 - \frac{1}{12}\varepsilon^2 k^4 \pm \dots}$

Explicit solution in Fourier space

$$\hat{u}^\varepsilon(k,t) \sim e^{\pm i[\omega(\varepsilon k)/\varepsilon]t}$$

Also for lattices there is a weakly dispersive limit equation for w^ε

Theorem (B.S. & F.Theil, 2018)

$$\sup_{t\leq T/\varepsilon^2} \|\partial_t \hat{w}^\varepsilon(.,t) - \partial_t \hat{u}^\varepsilon(.,t)\|_{L^2(\mathbb{R}^d)} \leq C\varepsilon$$

Part 2: Profile equations

Wave in heterogeneous media

$$\partial_t^2 u^\varepsilon = \nabla \cdot (a_\varepsilon(x) \nabla u^\varepsilon(x))$$

Weakly dispersive wave equation

$$\partial_t^2 w^\varepsilon = A D^2 w^\varepsilon + \varepsilon^2 E D^2 \partial_t^2 w^\varepsilon$$

Fourier representation

$$\hat{u}^\varepsilon(k,t) \sim e^{\pm i[\omega(\varepsilon k)/\varepsilon]t}$$

Profile equation

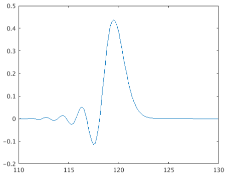
$$\partial_\tau V(z,\tau;q) = b(q)\, \partial_z^3 V$$

Profile in 1D: Linearized KdV equation

Task: Study the long time behavior of

$$\partial_t^2 u^\varepsilon = c^2 \partial_x^2 u^\varepsilon + \varepsilon^2 B\, \partial_x^4 u^\varepsilon$$

We want an equation for *this* $\longrightarrow$



Solution at time $t = 120$

Ansatz $u^\varepsilon(x,t) = V(x - ct, \varepsilon^2 t)$. Inserting yields

$$c^2 \partial_z^2 V - 2c\,\varepsilon^2 \partial_z \partial_\tau V + O(\varepsilon^4) = c^2 \partial_z^2 V + \varepsilon^2 B\, \partial_z \partial_z^3 V$$

Profile evolution equation: Linearized KdV equation; direction $q = \pm 1$:

$$\partial_\tau V^\varepsilon(z,\tau;q) = b(q)\, \partial_z^3 V^\varepsilon(z,\tau;q)$$

Initial condition:

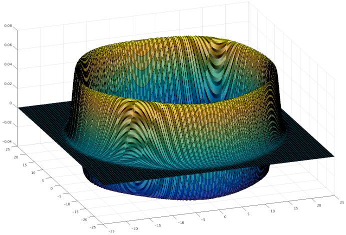
$$\hat{V}_0^\varepsilon(\xi;\pm 1) := \begin{cases} \hat{u}_0^\varepsilon(\pm\xi) & \text{for } \xi > 0 \\ 0 & \text{else} \end{cases}$$

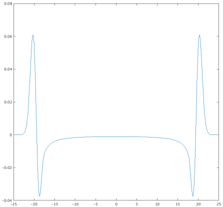
Reconstruction:

$$v^\varepsilon(x,t) := \begin{cases} V^\varepsilon\left(|x| - ct, \varepsilon^2 t; +1\right) & \text{for } x > 0 \\ V^\varepsilon\left(|x| - ct, \varepsilon^2 t; -1\right) & \text{for } x < 0 \end{cases}$$

Theorem (Lattice solution and profile solution)

$$\left\| \hat{u}^{\varepsilon}(.,\tau/\varepsilon^2) - \hat{v}^{\varepsilon}(.,\tau/\varepsilon^2) \right\|_{L^2(\mathbb{R})} \to 0 \quad \textit{as } \varepsilon \to 0$$





The profile is given by a linearized KdV equation

$$\partial_\tau V^{\varepsilon}(z,\tau;q) = b(q)\, \partial_z^3 V^{\varepsilon}(z,\tau;q)$$

Initial condition

$$\hat{V}_0^{\varepsilon}(\xi;q) = \begin{cases} \sqrt{\frac{\xi}{2\pi i}}\ \hat{u}_0^{\varepsilon}(\xi q) & \text{for } \xi > 0 \\ 0 & \text{else} \end{cases}$$

Reconstruction

$$v^{\varepsilon}(x,t) = \frac{1}{|x|^{1/2}} V^{\varepsilon}\left(|x| - ct, \varepsilon^2 t; \frac{x}{|x|} \right)$$

General functions in Fourier space

- Wave equation

$$\hat{u}^\varepsilon(k,t) = \hat{u}_0(k) e^{-ic|k|t}$$

- Lattice

$$\hat{u}^\varepsilon(k,t) = \hat{u}_0(k) e^{-ic|k|t} e^{i\frac{b}{2}|k|^3\varepsilon^2 t}$$

- Periodic media

$$\hat{u}^\varepsilon(k,t) = \hat{u}_0(k) e^{-i|k|_A t} e^{ib(k)\varepsilon^2 t}$$

Main result:

multiplication $\quad \hat{u}_0(k) \mapsto e^{ic|k|\tau/\varepsilon^2} e^{-ib(|k|)\tau} \hat{u}_0(k)$

is like application of $\quad \hat{\mathcal{Q}}_b^\rho = \mathcal{F}_d \circ \mathcal{S} \circ \mathcal{F}_1^{-1} \circ J_b \circ \mathcal{R}_\rho$

Theorem (A.Lamacz & B.S., 2020)

Let u_0 be initial data, $d \leq 3$, $b : \mathbb{R} \to \mathbb{R}$ a dispersion function (appropriate). For $\tau > 0$ and $k \in \mathbb{R}^d$ with $|k| > \rho$, as $\varepsilon \to 0$:

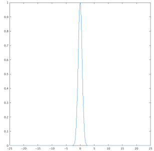
$$(\hat{\mathcal{Q}}_b^\rho \hat{u}_0)(k, \tau/\varepsilon^2) - e^{ic|k|\tau/\varepsilon^2} e^{-ib(|k|)\tau} \hat{u}_0(k) \to 0$$

Abstract reconstruction operators

Dimension $d \in \{1,2,3\}$, direction $q \in S^{d-1} \subset \mathbb{R}^d$

Restriction operator $\mathcal{R}$: For $u_0 = u_0(x)$

$$\hat{V}_0(\xi;q) := (\mathcal{R}\hat{u}_0)(\xi;q) := \left(\frac{|\xi|}{2\pi i}\right)^{(d-1)/2} \mathbf{1}_{\{\xi>0\}}\ \hat{u}_0(\xi q)$$

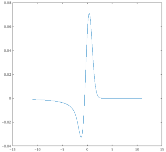


Cut of u_0 and of $\hat{u}_0$

Profile variables: ξ dual to z

Multiplication operator J_b: For $b : \mathbb{R} \to \mathbb{R}$ set

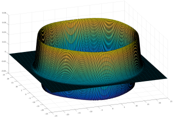
$$\hat{V}(\xi,\tau;q) := (J_b\hat{V}_0)(\xi,\tau;q) := e^{-ib(\xi)\tau}\hat{V}_0(\xi;q)$$



$V_0(.;q)$ and $V(.,\tau;q)$

Shell operator $\mathcal{S}$: For a profile $V = V(z,\tau;q)$

$$(\mathcal{S}V)(x,t) := \frac{1}{(ct)^{(d-1)/2}}\mathbf{1}_{\{|x|<2ct\}}\ V\left(|x| - ct, \varepsilon^2 t; \frac{x}{|x|}\right)$$



$\mathcal{S}(V)(.,t)$

The reconstruction operator Q_b:

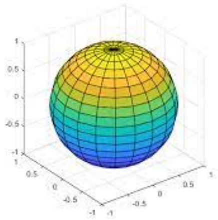
$$\mathcal{Q}_b = \mathcal{S} \circ \mathcal{F}_1^{-1} \circ J_b \circ \mathcal{R}$$

Stationary phase

Key in the proof: Write the reconstruction

$$v(x,t) := \frac{1}{(ct)^{(d-1)/2}} \mathbf{1}_{\{|x|<2ct\}} \; V\left(|x| - ct, \varepsilon^2 t; \frac{x}{|x|}\right)$$

in Fourier space



Lemma (Stationary phase)

$d \in \{1,2,3\}$ and $\kappa \in S^{d-1}$ a point. Test-function $\phi \in C^1(S^{d-1};\mathbb{R})$ supported in half sphere $\{q \in S^{d-1} \,|\, q \cdot \kappa \geq 0\}$. Then, as $N \to \infty$:

$$(2\pi i)^{-(d-1)/2} \int_{S^{d-1}} N^{(d-1)/2} \, e^{i(1-q\cdot\kappa)\,N} \, \phi(q) \, dS(q) \longrightarrow \phi(\kappa)$$

Results

- **Dispersion:** Can appear in ε-heterogeneous wave equations at time scale $t \sim \varepsilon^{-2}$

  A one-dimensional limit equation can be $\partial_t^2 w^\varepsilon = \partial_x^2 w^\varepsilon + \varepsilon^2 \partial_x^2 \partial_t^2 w^\varepsilon$
- **Profile equation:** Linearized KdV equation $\partial_\tau V = b\,\partial_z^3 V$
- **Abstract reconstruction:** Quite general evolutions in Fourier space can be approximated with profiles

Thank you!

[1] A.Lamacz: Dispersive effective models for waves in heterogeneous media. *Math. Models Methods Appl. Sci. 21(9)*, 2011

[2] T.Dohnal, A.Lamacz, B.Schweizer: Bloch-wave homogenization on large time scales and dispersive effective wave equations. *Multiscale Model. Simul. 12(2)*, 2014

[3] T.Dohnal, A.Lamacz, B.Schweizer: Dispersive homogenized models and coefficient formulas for waves in general periodic media. *Asymptot. Anal. 93(1-2)*, 2015

[4] B.Schweizer, F.Theil: Lattice dynamics on large time scales and dispersive effective equations, *SIAM J. Appl. Math. 78(6)*, 2018

[5] A.Lamacz, B.Schweizer: Representation of solutions to wave equations with profile functions, *Analysis and Applications*, 2020