

Effective Helmholtz problem in a domain with a Neumann sieve perforation

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Abstract: A first order model for the transmission of waves through a sound-hard perforation along an interface is derived. Mathematically, we study the Neumann problem for the Helmholtz equation in a complex geometry, the domain contains a periodic array of inclusions of size $\varepsilon > 0$ along a co-dimension 1 manifold. We derive effective equations that describe the limit as $\varepsilon \rightarrow 0$. At leading order, the Neumann sieve perforation has no effect; the corrector is given by a Helmholtz equation on the unperturbed domain with jump conditions across the interface. The corrector equations are derived with unfolding methods in L^1 -based spaces.

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1 Introduction

Our aim is to describe the transmission of waves through a Neumann sieve. Studies of this and related homogenization problems are available since the 1980ies, for early contributions see [18, 7]. Many different boundary conditions and many different scalings can be studied, which led to a large body of literature in this field. Our setting is very elementary: We study the homogeneous Neumann problem with ε -size inclusions that are situated with ε -periodicity along a manifold of co-dimension 1. We provide effective equations with a direct proof, analyzing quite general settings in arbitrary dimension.

We fix a frequency $\omega > 0$ and a source $f \in L^2(\Omega)$ and study the Helmholtz equation

$$-\Delta p^\varepsilon = \omega^2 p^\varepsilon + f \quad \text{in } \Omega_\varepsilon. \quad (1.1)$$

We may supplement (1.1) with a homogeneous Dirichlet boundary condition on $\partial\Omega$, we always use homogeneous Neumann boundary condition on $\partial\Omega_\varepsilon \setminus \partial\Omega$.

The domain Ω_ε is constructed from a Lipschitz domain $\Omega \subset \mathbb{R}^d$ by removing inclusions of size $\varepsilon > 0$ along a $(d-1)$ -dimensional manifold Γ_0 . More precisely, we set $\Omega_\varepsilon := \Omega \setminus \Sigma_\varepsilon$, where Σ_ε is the disjoint union of small obstacles. The number of

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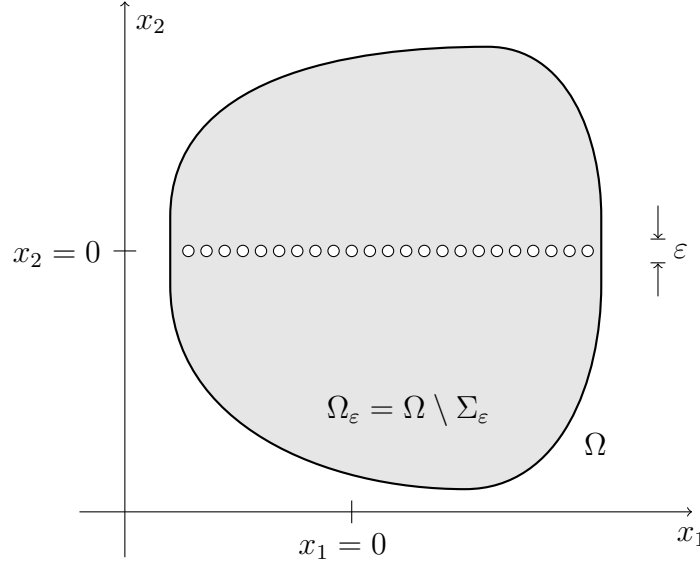


Figure 1: The Neumann sieve. The domain Ω is perforated along a lower dimensional manifold. The perforated domain Ω_ε (gray) is obtained by removing the union Σ_ε of obstacles. The Helmholtz equation is solved on Ω_ε with a homogeneous Neumann condition on $\partial\Sigma_\varepsilon$.

obstacles is of the order $\varepsilon^{-(d-1)}$, the single obstacle is denoted by Σ_k^ε , where $k \in \mathbb{Z}^{d-1}$ is used to number the obstacles. Each obstacle Σ_k^ε has a diameter of order ε and is obtained as a scaled copy of a Lipschitz domain $\Sigma \subset (-\frac{1}{2}, \frac{1}{2})^d$. The obstacles are periodically distributed along $\Gamma_0 \subset \Omega$, see Figure 1.

Let $p \in L^2(\Omega)$ be a weak limit of the solution sequence p^ε (after a trivial extension of the solution sequence across the obstacles). The effective problem in leading order is to determine an equation for p . The result for the leading order problem is trivial in the sense that the effect of the perforation gets lost: The equation for p is the Helmholtz equation

$$-\Delta p = \omega^2 p + f \quad \text{in } \Omega. \quad (1.2)$$

For a proof see Theorem 1 of [11].

The fact that, to leading order, the perforation has no effect, is well-known to experts in the field. Let us take an analytical perspective and sketch why the above result is quite clear: The solution sequence p^ε is bounded in $H^1(\Omega_\varepsilon)$ and can be extended to a bounded sequence in $H^1(\Omega)$. A weak limit of this sequence lies in $H^1(\Omega)$ and has therefore no jump across Γ_0 . On the other hand, no source is introduced by the inclusions, since homogeneous Neumann boundary conditions are used. This leads to the fact that also the normal derivative of p has no jump across Γ_0 . The combination of these two continuity statements yields that p solves the Helmholtz equation in the whole domain Ω .

Interesting equations occur at first order. For p^ε solving (1.1) and p solving (1.2) we study the corrector

$$v^\varepsilon := \frac{p^\varepsilon - p}{\varepsilon} \quad (1.3)$$

and weak limits v of the sequence v^ε (in appropriate function spaces). Since p^ε and

p solve the Helmholtz equation on the domain $\Omega \setminus \Gamma_0$, also v solves the equation on $\Omega \setminus \Gamma_0$. The main result of the work at hand concerns the jump conditions for v across Γ_0 . We denote a normal vector on Γ_0 by ν . We obtain that the jump $[v]$ is given in terms of ∇p along Γ_0 . Similarly, the jump $[\partial_\nu v]$ is given in terms of second derivatives of p , see (1.10).

Let us make a remark concerning the structure of the limiting system. An approximation of the solution p^ε for a fixed parameter $\varepsilon > 0$ is given by $p + \varepsilon v$. This quantity can be calculated in two steps, since the equations for p and v are decoupled: In a first step, the zero order approximation p can be calculated by solving (1.2), i.e. by neglecting the Neumann sieve. The first order approximation v can be calculated on the domain Ω (and not in a complex geometry) by solving (1.10). The jump conditions for v across Γ_0 are determined by p .

The same limit equations have been obtained in [9] in two dimensions and in a radially symmetric setting. Our result is valid in arbitrary dimension, the constructions allow to treat very general geometries and general conditions on the macroscopic boundaries. An important feature of our contribution is that we split the derivation of the effective system into two parts: In the two Sections 3 and 4 we derive the effective system under the assumption that the solution sequence v^ε is bounded in $W^{1,1}(\Omega_\varepsilon)$. In Section 5 we present sufficient conditions on the domain and the macroscopic boundary conditions such that the $W^{1,1}$ -boundedness assumption on v^ε is satisfied. This splitting of the proof makes each piece quite accessible. Furthermore, we can formulate the following loose interpretation: Whenever a limit system can be derived, it must have the form (1.10) of Theorem 1.2.

The literature on the subject is discussed in Section 1.2 below.

1.1 Setting and main result

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with Lipschitz boundary, containing the origin. Let the obstacle shape Σ be a Lipschitz domain $\Sigma \subset (-\frac{1}{2}, \frac{1}{2})^d$ such that the complement $(-\frac{1}{2}, \frac{1}{2})^d \setminus \Sigma$ is connected. We restrict to obstacles of maximal height 1 for notational convenience; the case $\Sigma \subset (-\frac{1}{2}, \frac{1}{2})^{d-1} \times (-M, M)$ for some $M > \frac{1}{2}$ is covered by a change of notation.

We use $k \in \mathbb{Z}^{d-1}$ to label the different obstacles and set

$$\Sigma_k^\varepsilon := \varepsilon (\Sigma + (k, 0)) \quad \text{for } k \in \mathbb{Z}^{d-1}, \quad I_\varepsilon := \{k \in \mathbb{Z}^{d-1} \mid \bar{\Sigma}_k^\varepsilon \subset \Omega\}. \quad (1.4)$$

The number of elements of I_ε is of order $\varepsilon^{-(d-1)}$. We denote the union of all obstacles as Σ_ε and denote the perforated domain as Ω_ε ,

$$\Sigma_\varepsilon := \bigcup_{k \in I_\varepsilon} \Sigma_k^\varepsilon, \quad \Omega_\varepsilon := \Omega \setminus \bar{\Sigma}_\varepsilon. \quad (1.5)$$

Let $n = n_\varepsilon(x)$ be the outer normal of Ω_ε on $\partial\Omega_\varepsilon$. In our construction, the perforation Σ_ε is located along the submanifold $\Gamma_0 := (\mathbb{R}^{d-1} \times \{0\}) \cap \Omega$. The interface $\Gamma_0 =: \Gamma \times \{0\}$ with $\Gamma \subset \mathbb{R}^{d-1}$ has the upward pointing normal $\nu = e_d$.

We assume that $f \in L^2(\Omega)$ is a given source. The weak form of (1.1) with the indicated boundary conditions is the following: We search for $p^\varepsilon \in \mathcal{H}_\varepsilon := \{q^\varepsilon \in$

$H^1(\Omega_\varepsilon)|_{q^\varepsilon=0 \text{ on } \partial\Omega}$ such that

$$\int_{\Omega_\varepsilon} \nabla p^\varepsilon \cdot \nabla \varphi = \int_{\Omega_\varepsilon} \omega^2 p^\varepsilon \varphi + \int_{\Omega_\varepsilon} f \varphi \quad \forall \varphi \in \mathcal{H}_\varepsilon. \quad (1.6)$$

In what follows, we denote by $\mathcal{P}_\varepsilon : L^2(\Omega_\varepsilon) \rightarrow L^2(\Omega)$ the extension operator that extends a function by 0 to all of Ω .

We will assume that ω^2 is not an eigenvalue of the operator $-\Delta$ on Ω under the imposed boundary conditions on $\partial\Omega$, in short: $\omega^2 \notin \sigma(-\Delta)$.

Theorem 1.1 (Theorem 1 of [11]). *Let $\Omega \subset \mathbb{R}^d$, ω with $\omega^2 \notin \sigma(-\Delta)$, $f \in L^2(\Omega)$, and Ω_ε be as above. Let p^ε be solutions to (1.6) and let the dimension be $d = 3$.*

With the unique weak solution $p \in H_0^1(\Omega)$ of (1.2) holds

$$\mathcal{P}_\varepsilon p^\varepsilon \rightarrow p \quad \text{and} \quad \mathcal{P}_\varepsilon \nabla p^\varepsilon \rightharpoonup \nabla p \quad \text{in } L^2(\Omega). \quad (1.7)$$

Let f have the regularity $H^1 \cap C^\alpha$, $\alpha \in (0, 1)$, in an open neighborhood of Γ_0 and let $\partial\Omega$ be of class C^3 in a neighborhood of $\bar{\Gamma}_0 \cap \partial\Omega$. For a constant $C = C(f)$ holds

$$\|p - \mathcal{P}_\varepsilon p^\varepsilon\|_{L^2(\Omega)} + \|\nabla p - \mathcal{P}_\varepsilon \nabla p^\varepsilon\|_{L^2(\Omega)} \leq C\varepsilon^{1/2}. \quad (1.8)$$

Theorem 1.1 establishes that the limit of the Neumann sieve solutions is given by the function p . In particular, the limit problem does not contain any contribution from the perforation. We remark that the claims of Theorem 1.1 remain true for $d \geq 2$, the proofs of [11] work in any dimension.

The aim of this contribution is to characterize the corrector v^ε that was defined in (1.3). Our main result is the derivation of the limit system (1.10), in which the jump $[v]$ of v and the jump $[\partial_\nu v]$ of its normal derivative across Γ_0 are used. Comments on the two expressions are given after the theorem. The theorem uses the following boundedness on the solution sequence as an assumption: For some $C > 0$, independent of ε , there holds

$$\|v^\varepsilon\|_{W^{1,1}(\Omega_\varepsilon)} \leq C. \quad (1.9)$$

This boundedness can be verified in relevant cases: Theorem 5.3 provides (1.9) in a periodic problem, Theorem 5.4 in a Dirichlet problem.

Different boundary conditions on $\partial\Omega$ can be treated. One choice is to consider homogeneous Dirichlet conditions in all problems: in equation (1.1) for p^ε , in (1.2) for p , and in (1.10) below for v .

Theorem 1.2 (Effective system for the corrector). *Let Ω and Ω_ε in dimension $d \geq 2$ be as above, let $\omega > 0$ satisfy $\omega^2 \notin \sigma(-\Delta)$. Let p^ε be solutions to (1.1) and let p be a solution to (1.2). We assume that $f \in L^2(\Omega)$ allows the regularity $p \in C^2$ in a neighborhood of Γ_0 and the convergence estimate (1.8). Let the correctors v^ε be given by (1.3), we assume the a priori bound (1.9).*

In this situation, $\mathcal{P}_\varepsilon v^\varepsilon \rightarrow v$ holds in the sense of measures on Ω and in $L_{\text{loc}}^1(\Omega)$, the limit function $v \in W^{1,1}(\Omega \setminus \Gamma_0)$ is characterized by the system:

$$\begin{aligned} -\Delta v &= \omega^2 v && \text{in } \Omega \setminus \Gamma_0, \\ [v] &= J \cdot \nabla p && \text{on } \Gamma_0, \\ [\partial_\nu v] &= \nabla \cdot (G \nabla p) && \text{on } \Gamma_0. \end{aligned} \quad (1.10)$$

Here, $G \in \mathbb{R}^{d \times d}$ and $J \in \mathbb{R}^d$ are given by cell problems, see (2.6) and (2.7).

Theorem 1.2 is a direct consequence of Proposition 4.3.

Remarks on jump quantities. For a piecewise continuous function v and $x \in \Gamma_0$, we set $[v](x) := \lim_{\delta \searrow 0} (v(x + \delta\nu) - v(x - \delta\nu))$. For $v \in H^1(\Omega \setminus \Gamma_0)$, the jump is well defined in the sense of traces. For piecewise C^1 functions v , the jump of the normal derivative $[\partial_\nu v]$ is defined accordingly. In our setting, the condition for $[\partial_\nu v]$ must be written in a weak form: First and third equation of (1.10) are written combined as

$$\int_{\Omega} \nabla v \cdot \nabla \varphi - \int_{\Omega} \omega^2 v \varphi = - \int_{\Gamma_0} [\nabla \cdot (G \nabla p)] \varphi \quad \forall \varphi \in C_c^\infty(\Omega). \quad (1.11)$$

On the right hand side, we can separate tangential and normal components, using the tangential derivatives $\tilde{\nabla} = (\partial_1, \dots, \partial_{d-1})$ on Γ_0 . With the tangential parts G_τ of G and J_τ of J as in (2.8), we can write

$$\nabla \cdot (G \nabla p) = \tilde{\nabla} \cdot (G_\tau \tilde{\nabla} p) + \tilde{\nabla} \cdot (J_\tau \partial_\nu p) - |\Sigma| \partial_\nu^2 p.$$

Assuming reflection symmetry for the obstacle shape Σ , the equations simplify further. We consider, for $1 \leq j \leq d-1$, the reflection $R_j : \mathbb{R}^d \rightarrow \mathbb{R}^d$, which is given as the linear function with $R_j e_j = -e_j$ and $R_j e_i = e_i$ for $i \neq j$.

Corollary 1.3 (Limit system for symmetric obstacles). *Let the obstacle shape Σ be invariant under all reflections R_j , $1 \leq j \leq d-1$. In this case, the effective coefficients are given by real coefficients $\beta_j > 0$ for $1 \leq j \leq d-1$ and $\gamma > |\Sigma|$ in the form $G = \text{diag}(\beta_1, \dots, \beta_{d-1}, -|\Sigma|)$ and $J = (0, \dots, 0, \gamma)$. The effective system (1.10) takes the form*

$$\begin{aligned} -\Delta v &= \omega^2 v && \text{in } \Omega \setminus \Gamma_0, \\ [v] &= \gamma \partial_\nu p && \text{on } \Gamma_0, \\ [\partial_\nu v] &= \sum_{j=1}^{d-1} \beta_j \partial_j^2 p - |\Sigma| \partial_\nu^2 p && \text{on } \Gamma_0. \end{aligned} \quad (1.12)$$

The corollary is shown in Section 4.2.

Remark on scalings. We study a very natural scaling of the geometry: the size of the obstacles and the periodicity are both of order $\varepsilon > 0$. The corrector v^ε is formally of order ε^{-1} , see its definition in (1.3) or the characterizing equations (3.1), which contain the factor ε^{-1} . As a consequence, $|\nabla v^\varepsilon|$ is of order ε^{-1} in the vicinity of the obstacles. Despite this fact, by decay properties away from the interfaces, the $L^1(\Omega_\varepsilon)$ -norms can be bounded; the work at hand is based on exactly this observation. We note that, in general, the $L^q(\Omega_\varepsilon)$ -norm of $|\nabla v^\varepsilon|$ is unbounded for every $q > 1$.

1.2 Literature

The literature offers two different ways to formulate effective limit equations. The analytical literature such as [9] or the contribution at hand derives limit models in the form that the first terms in the expansion of the solution in the small parameter are calculated. In the more applied literature, one finds effective equations which use a transmission impedance parameter. The transformation of the first system to the second system is part of the work in [9], the properties of the limit model in the context of electromagnetism are studied in [10].

Let us compare [9] to the work at hand. The authors of [9] treat the Helmholtz equation in two dimensions with a perforation along a circle. The solution is expanded in powers of δ (δ is our ε). Equation (118) in that analysis determines the first order corrector u_1 (our v) by a system in which the lowest order solution u_0 (our p) prescribes jump conditions. Our results can be compared with their main result, formulated as Proposition 11, which provides $u^\delta \approx u_0 + \delta u_1$. The proof is based on asymptotic expansions to second order. The fact that the solution can be expanded provides both, estimates and approximation properties. As indicated above, we take a quite different route here: Assuming that bounds are satisfied, we obtain the limit equation by tools that are close to two-scale convergence and periodic unfolding. In order to obtain the bounds, we construct approximate solutions in the spirit of asymptotic expansions – but the expansions can be of lower order.

The preceding work [11] introduced the approach that we use here: Convergence of v^ε and ∇v^ε in the sense of measures is assumed and equations for the limiting measures are extracted. The second theorem of [11] works with the assumption $\mu = \alpha[v]\nu \mathcal{H}^{d-1}$ for some α (the limit measure μ is introduced in Section 4). The work at hand does not yield this relation: In the case of symmetric inclusions we do find $\nu \cdot \mu = |\Sigma| \partial_\nu p \mathcal{H}^{d-1} = (|\Sigma|/\gamma)[v] \mathcal{H}^{d-1}$, which suggests $\alpha = |\Sigma|/\gamma$, but the tangential components of μ are, in general, not vanishing.

A study where the obstacles are replaced by highly oscillatory coefficients can be found in [6]. In [15], a parabolic problem within the separating layer is studied; a non-trivial lowest order behavior is obtained by an appropriate scaling of the coefficients of the parabolic equation. The article [5] is devoted to unfolding methods to analyze more general transmission problems where the obstacle scaling can be different from the periodicity. The authors are particularly interested in Dirichlet obstacles that are so small compared to the periodicity ε that a non-trivial effective description for the interface occurs. The method of Γ -convergence for functional of elasticity in the nonlinear Neumann sieve problem are studied in [1].

A quite different perspective is taken in [2]. The paper contains the statement about the vanishing effect of the perforation for $\varepsilon \rightarrow 0$, but the main focus is on radiation conditions in the case of unbounded domains and in transmission losses for coated inclusions.

Besides [9] and [6], also the following works use asymptotic expansions. Electromagnetic scattering is studied in [10] and [8]. The work [17] deals essentially with our setting. In [20], the authors are interested, in particular, in the end-points of the perforation. We recall at this point the loose interpretation of our result: We show that our effective equations (1.10) hold whenever a priori estimates are satisfied. In this sense, our equations are also valid in geometries in which the perforation ends inside the domain.

There are many works in which related geometries are studied: [3] analyzes Robin boundary conditions in cases where the scale of the inclusion is smaller than ε ; nontrivial effective equations are obtained for appropriate (exponential) scalings. For the case that there is no perforation inside the domain, but rather the boundary $\partial\Omega$ is perturbed in a periodic fashion, results are available in [14]: As in our case, the lowest order approximation of the solution is given by a trivial limit problem, the first order corrector solves a modified macroscopic problem.

Resonant inclusions are studied, e.g. in [16] and [13]. Resonant perturbations of $\partial\Omega$ are described in terms of the spectrum of the perturbed domain in [4]. Resonant inclusions in the entire domain are studied in [12], see [19] for an overview.

2 Cell-problem

The periodicity cell is a cylinder that contains a single inclusion. We set

$$Y := \left(-\frac{1}{2}, \frac{1}{2}\right)_{\text{per}}^{d-1} \times \mathbb{R}, \quad Z := Y \setminus \Sigma, \quad (2.1)$$

and recall that we assumed that $\Sigma \subset \left(-\frac{1}{2}, \frac{1}{2}\right)^d$ is a compactly contained Lipschitz domain. The subscript “per” indicates that all functions on Y are understood as functions on the flat torus; solutions to elliptic equations on Z must be periodic on ∂Y and the equation must be satisfied on ∂Y , which means that also derivatives are periodic in the sense of traces.

The cell problem is a system of equations for a function $w \in H_{\text{loc}}^1(Z)$, it uses a parameter $\xi \in \mathbb{R}^d$, and reads

$$\begin{aligned} -\Delta w &= 0 && \text{in } Z, \\ \partial_n w &= n \cdot \xi && \text{on } \partial\Sigma, \end{aligned} \quad (2.2)$$

where $n : \partial\Sigma \rightarrow \mathbb{R}^d$ is the exterior normal of Z .

Lemma 2.1 (Existence and uniqueness for cell problem). *For every $\xi \in \mathbb{R}^d$, there exists a solution w to problem (2.2) in the space*

$$w \in \dot{H}(Z) := \{w \in H_{\text{loc}}^1(Z) \mid \nabla w \in L^2(Z)\}, \quad (2.3)$$

$$\|w\|_{\dot{H}}^2 := \int_{Z \cap \{|y_d| < 1\}} |w|^2 + \int_Z |\nabla w|^2. \quad (2.4)$$

We recall the periodicity of functions $w \in \dot{H}(Z)$ in lateral directions is imposed by definition of Z . The solution w is unique up to additive constants.

Proof. We consider the functional

$$I(w) := \frac{1}{2} \int_Z |\nabla w|^2 - \int_{\partial\Sigma} \xi \cdot n w \quad (2.5)$$

on $\dot{H}(Z)$, restricted to the subspace of functions with vanishing integral over the set $Z \cap \{|y_d| < 1\}$. The functional I is coercive by the Poincaré inequality. The direct method can be applied and provides a solution w to the cell problem. Uniqueness on the chosen subspace follows from strict convexity. $\square$

With the help of the cell problem, we define a coefficient matrix $G \in \mathbb{R}^{d \times d}$; the name indicates that *gradients* are averaged. We set, for arbitrary $\xi \in \mathbb{R}^d$ and a solution $w = w_\xi$ of (2.2),

$$G \xi := \int_Z \nabla w \in \mathbb{R}^d. \quad (2.6)$$

We note that, if the integral is well defined, the right hand side is a vector in $\mathbb{R}^d$ that depends linearly on ξ . Therefore, (2.6) can define a matrix $G \in \mathbb{R}^{d \times d}$.

We next define a coefficient vector $J \in \mathbb{R}^d$; the name indicates that a *jump-value* of w is extracted. Since we are in a rescaled setting, the jump-value is given by the difference of values of w at $\pm\infty$. For arbitrary $\xi \in \mathbb{R}^d$ we use $w = w_\xi$ and set

$$J \cdot \xi := - \lim_{\zeta \rightarrow \infty} \int_{\{y_d = \zeta\}} w + \lim_{\zeta \rightarrow -\infty} \int_{\{y_d = \zeta\}} w \in \mathbb{R}. \quad (2.7)$$

Lemma 2.2. *The matrix G and the vector J are well defined. They have the form*

$$G = \begin{pmatrix} G_\tau & J_\tau \\ 0 & -|\Sigma| \end{pmatrix}, \quad J = \begin{pmatrix} J_\tau \\ \gamma \end{pmatrix}, \quad (2.8)$$

for a symmetric and positive definite matrix $G_\tau \in \mathbb{R}^{(d-1) \times (d-1)}$, a vector $J_\tau \in \mathbb{R}^{d-1}$, and a number $\gamma \in \mathbb{R}$ with $\gamma > |\Sigma|$.

Proof. Step 1: Properties of ∇w . Each derivative $\partial_i w$ is a harmonic function in $Z \cap \{y_d > 1/2\}$. We consider, for $|\zeta| > 1/2$, and $i \in \{1, \dots, d\}$, the averages $a_i(\zeta) := \int_{\{y_d = \zeta\}} \partial_i w$. The averages $a_i(\zeta)$ vanish for $i \leq d-1$ by periodicity of w . The average $a_d(\zeta) := \int_{\{y_d = \zeta\}} \partial_d w$ satisfies $\partial_\zeta a_d(\zeta) = \int_{\{y_d = \zeta\}} \partial_d^2 w = 0$ because of $\Delta w = 0$ and periodicity. We find that $a_d(\zeta)$ is independent of ζ . On the other hand, because of $w \in \dot{H}(Z)$, the function $a_d(\cdot)$ is square integrable. It therefore vanishes identically for $|\zeta| \geq 1/2$.

Step 2: G and J are well defined. Since averages along horizontal planes (not intersecting Σ) vanish, we see that the definition of G in (2.6) is equivalent to

$$G \xi := \int_{(-\frac{1}{2}, \frac{1}{2})^d \setminus \Sigma} \nabla w.$$

This is a well defined element of $\mathbb{R}^d$.

The fact that $a_d(\zeta)$ vanishes for $\zeta \geq 1$ implies that the averages of the values, $b(\zeta) := \int_{\{y_d = \zeta\}} w$, is independent of ζ for $\zeta \geq 1$. This implies that J is well-defined and indeed identical with the number

$$J \cdot \xi = -b(1/2) + b(-1/2) = - \int_{\{y_d = 1/2\}} w(y) + \int_{\{y_d = -1/2\}} w(y).$$

Step 3: Properties of G . We define the “tangential” part of the matrix G as the sub-matrix $G_\tau \in \mathbb{R}^{(d-1) \times (d-1)}$, $(G_\tau)_{i,j} = G_{i,j}$ for $i, j \leq d-1$.

Let w be a cell solution for $\xi \in \mathbb{R}^d$. We consider the function $F : Y \rightarrow \mathbb{R}^d$,

$$F(y) := \begin{cases} \nabla w(y) & \text{for } y \in Z \\ \xi & \text{else.} \end{cases} \quad (2.9)$$

By the properties of w , the function F has a vanishing divergence in Y . As noted in Step 1, averages of F over planes $\{y_d = \zeta\}$ vanish for $|\zeta| > 1/2$. The fact that F

has a vanishing divergence implies that integrals of the component F_d over planes $\{y_d = \zeta\}$ vanish for every $\zeta \in \mathbb{R}$. This allows to calculate, for $\xi = e_d$,

$$0 = \int_Y F_d = \int_\Sigma F_d + \int_Z F_d = \int_\Sigma 1 + \int_Z e_d \cdot \nabla w = |\Sigma| + G_{d,d},$$

which provides $G_{d,d} = -|\Sigma|$.

We consider now $\xi = e_i$ with $i < d$ and the corresponding flux function F of (2.9). We calculate as above and find

$$0 = \int_Y F_d = \int_\Sigma F_d + \int_Z F_d = \int_\Sigma e_d \cdot e_i + \int_Z e_d \cdot \nabla w = 0 + G_{d,i},$$

which provides $G_{d,i} = 0$.

Let now w_i be a cell solution for $\xi = e_i$ and w_j be a cell solution for $\xi = e_j$. In the case $j < d$ we can calculate

$$\begin{aligned} G_{j,i} &= e_j \cdot \int_Z \nabla w_i = \int_Z \nabla \cdot (w_i e_j) = \int_{\partial\Sigma} n \cdot (w_i e_j) = \int_{\partial\Sigma} n \cdot e_j w_i \\ &= \int_{\partial\Sigma} n \cdot \nabla w_j w_i = \int_Z \nabla w_j \cdot \nabla w_i. \end{aligned}$$

Since the right hand side is symmetric in i and j , the matrix G_τ is symmetric. The formula also implies that G_τ is positive definite.

We can use the above calculation also to find some information on $G_{j,d}$, $j < d$. With the cell solutions w_j and w_d to $\xi = e_j$ and $\xi = e_d$ we calculate

$$\begin{aligned} G_{j,d} &= \int_Z \nabla w_j \cdot \nabla w_d = \int_{\partial\Sigma} w_j n \cdot \nabla w_d = \int_{\partial\Sigma} w_j n \cdot e_d \\ &= \int_Z \nabla \cdot (w_j e_d) - \int_{\{y_d=1/2\}} w_j + \int_{\{y_d=-1/2\}} w_j \\ &= G_{d,j} + J \cdot e_j = J \cdot e_j. \end{aligned}$$

We can therefore write G as in (2.8). Concerning the value of γ , we calculate

$$\begin{aligned} -|\Sigma| &= G_{d,d} = \int_{Z \cap \{|y_d| < 1/2\}} e_d \cdot \nabla w_d \\ &= \int_{\{y_d=1/2\}} w_d - \int_{\{y_d=-1/2\}} w_d + \int_{\partial\Sigma} w_d n \cdot e_d \\ &= -J \cdot e_d + \int_{\partial\Sigma} w_d n \cdot \nabla w_d = -\gamma + \int_Z |\nabla w_d|^2. \end{aligned}$$

This provides $\gamma > |\Sigma|$ and concludes the characterization of G and J . $\square$

Lemma 2.3. *Let w be the cell solution to $\xi \in \mathbb{R}^d$. There holds*

$$e_j \cdot \int_{\partial\Sigma} n w = \begin{cases} e_j \cdot G \xi & \text{for } j < d, \\ e_d \cdot G \xi + J \cdot \xi & \text{for } j = d. \end{cases} \quad (2.10)$$

Proof. The claim follows with an integration by parts. For $j = d$ we calculate

$$e_d \cdot G \xi = e_d \cdot \int_Z \nabla w = -J \cdot \xi + e_d \cdot \int_{\partial\Sigma} n w.$$

For $j < d$ one calculates accordingly (without a jump contribution). $\square$

3 Elementary unfolding operations

In the setting of Theorem 1.2, we are given a function $p \in H_0^1(\Omega)$, which solves a Helmholtz equation with a right hand side $f \in C^\alpha \cap H^1$. By elliptic regularity, p is of class H^3 and of class $C^{2,\alpha}$ in a neighborhood of Γ_0 . Our aim is to analyze limits of solutions v^ε to the following corrector problem:

$$\begin{aligned} -\Delta v^\varepsilon &= \omega^2 v^\varepsilon && \text{in } \Omega_\varepsilon, \\ \partial_n v^\varepsilon &= -\varepsilon^{-1} n \cdot \nabla p && \text{on } \partial\Sigma_\varepsilon, \\ v^\varepsilon &= 0 && \text{on } \partial\Omega. \end{aligned} \quad (3.1)$$

System (3.1) has the following weak formulation: The function $v^\varepsilon \in H^1(\Omega_\varepsilon)$ satisfies $v^\varepsilon|_{\partial\Omega} = 0$ and

$$\int_{\Omega_\varepsilon} \nabla v^\varepsilon \cdot \nabla \varphi = - \int_{\partial\Omega_\varepsilon} \frac{1}{\varepsilon} n \cdot \nabla p \varphi + \int_{\Omega_\varepsilon} \omega^2 v^\varepsilon \varphi \quad \forall \varphi \in H_0^1(\Omega). \quad (3.2)$$

In this and the next section, we derive a limit system for v^ε .

3.1 Estimates for rescaled averaged functions

We use the cell problem domain $Z := Y \setminus \Sigma$ as in the last section and, for $M > 1$, the truncated domain $Z_M := Z \cap \{|y_d| < M\}$.

Lemma 3.1 (Estimates for rescaled averaged functions). *Let v^ε be a sequence of functions $v^\varepsilon : \Omega_\varepsilon \rightarrow \mathbb{R}$ with the property*

$$\|v^\varepsilon\|_{L^2(\Omega_\varepsilon)} + \|\nabla v^\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq C_0 \varepsilon^{-1/2}. \quad (3.3)$$

Let φ be a test function, $\varphi \in C_c^\infty(\Omega)$. We introduce a rescaled averaged function on Z by setting, for $y \in Z$,

$$V_\varphi^\varepsilon(y) := \frac{1}{|I_\varepsilon|} \sum_{k \in I_\varepsilon} v^\varepsilon(\varepsilon(k+y)) \varphi(\varepsilon(k+y)). \quad (3.4)$$

Then the function V_φ^ε satisfies, with $C_1 = C_1(C_0)$ and $C_2 = C_2(M, C_0)$ independent of ε , the estimates

$$\int_Z |\nabla_y V_\varphi^\varepsilon|^2 \leq C_1, \quad (3.5)$$

$$\int_{Z_M} |V_\varphi^\varepsilon|^2 \leq C_2 \varepsilon^{-1}. \quad (3.6)$$

Proof. Step 1: Estimate (3.5). The gradient can be estimated with a direct calculation, using an arbitrary number $M > 1$ and Jensen's inequality in the second line.

$$\begin{aligned} \int_{Z_M} |\nabla_y V_\varphi^\varepsilon|^2 &= \int_{Z_M} \left| \frac{1}{|I_\varepsilon|} \sum_{k \in I_\varepsilon} \nabla_y [v^\varepsilon(\varepsilon(k+y)) \varphi(\varepsilon(k+y))] \right|^2 dy \\ &\leq \frac{1}{|I_\varepsilon|} \sum_{k \in I_\varepsilon} \int_{Z_M} |\nabla_y [v^\varepsilon(\varepsilon(k+y)) \varphi(\varepsilon(k+y))]|^2 dy \\ &\leq \frac{1}{|I_\varepsilon|} \sum_{k \in I_\varepsilon} \int_{Z_M} \varepsilon^2 |(\nabla_x v^\varepsilon \varphi)(\varepsilon(k+y)) + (v^\varepsilon \nabla_x \varphi)(\varepsilon(k+y))|^2 dy \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{|I_\varepsilon|} \sum_{k \in I_\varepsilon} \int_{\varepsilon(k+Z_M)} \varepsilon^2 |(\nabla_x v^\varepsilon \varphi)(x) + (v^\varepsilon \nabla_x \varphi)(x)|^2 \varepsilon^{-d} dx \\
&= \frac{\varepsilon^{2-d}}{|I_\varepsilon|} \int_{\Gamma \times (-\varepsilon M, \varepsilon M)} |(\nabla_x v^\varepsilon \varphi)(x) + (v^\varepsilon \nabla_x \varphi)(x)|^2 dx \\
&\leq C\varepsilon \int_{\Gamma \times (-\varepsilon M, \varepsilon M)} |\nabla_x v^\varepsilon|^2 + |v^\varepsilon|^2 dx \leq C_1
\end{aligned}$$

by (3.3). The constant is independent of M , we therefore obtain (3.5). To be formally correct in the above calculation, we agree that all functions are extended by zero to the outside of Ω .

Step 2: Estimate (3.6). We first investigate a boundary integral. We estimate the integral of the function V_φ^ε over the upper boundary of Z_M , the set $\{y_d = M\}$.

$$\begin{aligned}
\left| \int_{Z \cap \{y_d = M\}} V_\varphi^\varepsilon(y) dy \right| &= \left| \int_{Z \cap \{y_d = M\}} \frac{1}{|I_\varepsilon|} \sum_{k \in I_\varepsilon} (v^\varepsilon \varphi)(\varepsilon(k+y)) dy \right| \\
&\leq C \left| \int_{\Omega \cap \{x_d = \varepsilon M\}} (v^\varepsilon \varphi)(x) dx \right| \leq C \|v^\varepsilon \varphi\|_{H^1(\Omega_\varepsilon)} \leq C\varepsilon^{-1/2}.
\end{aligned}$$

In the last line we used a trace theorem for the domain $\Omega \cap \{x_d > \varepsilon M\}$.

Step 1 provides a bound for the $L^2(Z_M)$ -norm of $\nabla_y V_\varphi^\varepsilon$. The Poincaré inequality on Z_M for functions with a controlled boundary integral implies (3.6). We exploit here that $Z = Y \setminus \Sigma$ is connected. $\square$

3.2 Limits of rescaled averaged functions

Lemma 3.2 (Limits of an averaged and rescaled version of v^ε). *Let v^ε be a sequence of solutions to (3.1) satisfying*

$$\|v^\varepsilon\|_{L^2(\Omega_\varepsilon)} + \|\nabla v^\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq C_0 \varepsilon^{-1/2}. \quad (3.7)$$

Let φ be a test function, $\varphi \in C_c^\infty(\Omega)$. We use the function $V_\varphi^\varepsilon : Z \rightarrow \mathbb{R}$ that was defined in (3.4). We set $V_{\varphi,0}^\varepsilon := V_\varphi^\varepsilon - c_\varepsilon$, where $c_\varepsilon \in \mathbb{R}$ is chosen in such a way that $V_{\varphi,0}^\varepsilon$ has a vanishing average in Z_1 . Then there holds

$$V_{\varphi,0}^\varepsilon \rightharpoonup w \text{ in } \dot{H}^1(Z) \quad (3.8)$$

as $\varepsilon \rightarrow 0$, where w is the cell-problem solution to

$$\xi := -\frac{1}{|\Gamma_0|} \int_{\Gamma_0} \nabla p \varphi \in \mathbb{R}^d. \quad (3.9)$$

Furthermore, there holds

$$e_j \cdot \int_{\partial \Sigma_\varepsilon} n v^\varepsilon \varphi \rightarrow |\Gamma_0| e_j \cdot \int_{\partial \Sigma} n w. \quad (3.10)$$

We note that the right hand side of (3.10) can be expressed with the help of (2.10). For $j < d$ we conclude from (3.10)

$$e_j \cdot \int_{\partial \Sigma_\varepsilon} n v^\varepsilon \varphi \rightarrow |\Gamma_0| e_j \cdot G \cdot \left(-\frac{1}{|\Gamma_0|} \int_{\Gamma_0} \nabla p \varphi \right) = - \int_{\Gamma_0} e_j \cdot G \nabla p \varphi. \quad (3.11)$$

For $j = d$ we conclude

$$e_d \cdot \int_{\partial \Sigma_\varepsilon} n v^\varepsilon \varphi \rightarrow \int_{\Gamma_0} |\Sigma| \partial_\nu p \varphi - \int_{\Gamma_0} J \cdot \nabla p \varphi. \quad (3.12)$$

Proof. Step 1. Bounds and weakly convergent subsequence. Lemma 3.1 can be applied. Estimate (3.5) implies that gradients of $V_{\varphi,0}^\varepsilon$ are bounded in $L^2(Z)$. Since the average of $V_{\varphi,0}^\varepsilon$ vanishes on Z_1 by construction, the sequence $V_{\varphi,0}^\varepsilon$ is bounded in $\dot{H}^1(Z)$. We select a subsequence $\varepsilon \rightarrow 0$ and a limit function w such that

$$V_{\varphi,0}^\varepsilon \rightharpoonup w \text{ in } \dot{H}^1(Z). \quad (3.13)$$

Step 2. Equations for w . Let ψ be a compactly supported C^∞ -function on Y . Since Y has a torus structure in the first components, this implies that the periodic extension of ψ is again a C^∞ -function. In the following, we identify ψ with its periodic extension in the lateral directions e_1 to e_{d-1} . By compactness of the support we can choose $M > 0$ large enough to have the support of ψ contained in $Y_M := Y \cap \{|y_d| < M\}$.

We calculate, in the limit $\varepsilon \rightarrow 0$,

$$\begin{aligned} \int_Z \nabla_y w \cdot \nabla_y \psi &\leftarrow \int_{Z_M} \nabla_y V_{\varphi,0}^\varepsilon \cdot \nabla_y \psi \\ &= \int_{Z_M} \frac{1}{|I_\varepsilon|} \sum_{k \in I_\varepsilon} \varepsilon [(\nabla_x v^\varepsilon \varphi)(\varepsilon(k+y)) + (v^\varepsilon \nabla_x \varphi)(\varepsilon(k+y))] \cdot \nabla_y \psi(y) dy \\ &= \frac{\varepsilon}{|I_\varepsilon| \varepsilon^d} \int_{\Gamma \times (-\varepsilon M, \varepsilon M)} \nabla_x v^\varepsilon(x) \varphi(x) \cdot \nabla_y \psi(x/\varepsilon) dx \\ &\quad + \int_{Z_M} \frac{1}{|I_\varepsilon|} \sum_{k \in I_\varepsilon} \varepsilon (v^\varepsilon \nabla_x \varphi)(\varepsilon(k+y)) \cdot \nabla_y \psi(y) dy \\ &= \frac{\varepsilon^{-(d-1)}}{|I_\varepsilon|} \int_{\Gamma \times (-\varepsilon M, \varepsilon M)} \varepsilon \nabla_x v^\varepsilon(x) \cdot \nabla_x (\varphi(x) \psi(x/\varepsilon)) dx \\ &\quad - \frac{\varepsilon^{-(d-1)}}{|I_\varepsilon|} \int_{\Gamma \times (-\varepsilon M, \varepsilon M)} \varepsilon \nabla_x v^\varepsilon(x) \cdot \nabla_x \varphi(x) \psi(x/\varepsilon) dx \\ &\quad + \int_{Z_M} \varepsilon \frac{1}{|I_\varepsilon|} \sum_{k \in I_\varepsilon} (v^\varepsilon \nabla_x \varphi)(\varepsilon(k+y)) \cdot \nabla_y \psi(y) dy. \end{aligned} \quad (3.14)$$

We claim that the second and third integral on the right hand side vanish in the limit $\varepsilon \rightarrow 0$. For the second term, this is clear because of $|I_\varepsilon| = O(\varepsilon^{-(d-1)})$ and $\varepsilon \|\nabla_x v^\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq C \varepsilon^{1/2}$. Concerning the third term, we observe that the function

$$F^\varepsilon(y) := \frac{1}{|I_\varepsilon|} \sum_{k \in I_\varepsilon} (v^\varepsilon \nabla_x \varphi)(\varepsilon(k+y))$$

is of the form that was analyzed in Lemma 3.1; we only have to interpret the derivatives of φ as new weights. Estimate (3.6) implies $\|F^\varepsilon\|_{L^2(Y_M)} \leq C\varepsilon^{-1/2}$. We use this to evaluate the last integral: Because of the factor ε , the integral containing F^ε vanishes in the limit $\varepsilon \rightarrow 0$.

We continue the calculation (3.14), keeping only the first integral and exploiting

$$|I_\varepsilon| \varepsilon^{d-1} \rightarrow |\Gamma_0|.$$

We use equation (3.2) for v^ε in the second equality to find

$$\begin{aligned} |\Gamma_0| \int_Z \nabla_y w \cdot \nabla_y \psi &= \lim_{\varepsilon \rightarrow 0} \int_{\Gamma \times (-\varepsilon M, \varepsilon M)} \varepsilon \nabla_x v^\varepsilon(x) \cdot \nabla_x (\varphi(x) \psi(x/\varepsilon)) dx \\ &= - \lim_{\varepsilon \rightarrow 0} \int_{\partial \Sigma_\varepsilon} n(x) \cdot \nabla p(x) \varphi(x) \psi(x/\varepsilon) d\mathcal{H}^{d-1}(x) \\ &= - \lim_{\varepsilon \rightarrow 0} \sum_{k \in I_\varepsilon} \varepsilon^{d-1} \int_{\partial \Sigma} n(y) \cdot (\psi(y) (\varphi \nabla p)(\varepsilon(k+y))) d\mathcal{H}^{d-1}(y) \\ &= - \int_{\Gamma_0} (\varphi \nabla p)(x) dx \cdot \int_{\partial \Sigma} n(y) \psi(y) d\mathcal{H}^{d-1}(y). \end{aligned}$$

The last equality follows from the Lipschitz continuity of φ and ∇p . We conclude that w solves the cell problem with ξ from (3.9).

Step 3. Convergence (3.10). We can perform a direct calculation, exploiting that an integral over $\partial \Sigma$ of the normal vector n (multiplied with a constant), vanishes.

$$\begin{aligned} \int_{\partial \Sigma_\varepsilon} n v^\varepsilon \varphi &= \sum_{k \in I_\varepsilon} \varepsilon^{d-1} \int_{\partial \Sigma} n(y) (v^\varepsilon \varphi)(\varepsilon(k+y)) d\mathcal{H}^{d-1}(y) \\ &= |I_\varepsilon| \varepsilon^{d-1} \int_{\partial \Sigma} n(y) V_{\varphi,0}^\varepsilon(y) d\mathcal{H}^{d-1}(y) \rightarrow |\Gamma_0| \int_{\partial \Sigma} n w \end{aligned}$$

by the trace theorem in $\dot{H}^1(Z)$. This shows the claim. $\square$

4 Effective equations

Lemma 4.1. *Let Ω_ε be as described in Section 1.1 and let v^ε be a sequence of solutions to (3.1) satisfying the $W^{1,1}(\Omega_\varepsilon)$ -bound (1.9). Then there exists a subsequence $\varepsilon \rightarrow 0$, a limit function $v \in W^{1,1}(\Omega \setminus \Gamma_0)$, and a measure $\mu \in \mathcal{M}(\Omega)$, $\text{supp}(\mu) \subset \Gamma_0$, such that the following convergences hold for $\varepsilon \rightarrow 0$:*

$$\mathcal{P}_\varepsilon v^\varepsilon \xrightarrow{*} v \quad \text{and} \quad \mathcal{P}_\varepsilon \nabla v^\varepsilon \xrightarrow{*} \nabla v + \mu \quad (4.1)$$

weakly- in the sense of measures on Ω , the first convergence also in $L^1_{\text{loc}}(\Omega)$.*

Proof. The sequences $\mathcal{P}_\varepsilon v^\varepsilon$ and $\mathcal{P}_\varepsilon \nabla v^\varepsilon$ are both bounded in $L^1(\Omega)$ by (1.9). By compactness in the space $\mathcal{M}(\Omega)$ of signed Borel measures on Ω , we find a subsequence $\varepsilon \rightarrow 0$ and limit measures $m, \nu \in \mathcal{M}(\Omega)$ such that $\mathcal{P}_\varepsilon v^\varepsilon \xrightarrow{*} m$ and $\mathcal{P}_\varepsilon \nabla v^\varepsilon \xrightarrow{*} \nu$. We decompose the limit measures in their absolute continuous and their singular part

with respect to the Lebesgue measure $\mathcal{L}^d$ as $m = v d\mathcal{L}^d + m_{\text{sing}}$ and $\nu = g d\mathcal{L}^d + \mu$. The densities v and g are of class $L^1(\Omega)$ by construction.

Let $\tilde{\Omega} \subset \Omega \setminus \Gamma_0$ be a compactly contained subset. The sequence v^ε is L^1 -bounded and solves a homogeneous elliptic equation in a neighborhood of $\tilde{\Omega}$. This implies that v^ε is H^2 -bounded in $\tilde{\Omega}$. The improved regularity of v^ε away from Γ_0 has two consequences: (i) m_{sing} and μ are supported on Γ_0 . (ii) $g = \nabla v$ in $\Omega \setminus \Gamma_0$. With these observations we have $v \in W^{1,1}(\Omega \setminus \Gamma_0)$ and the statement on the support of μ .

It remains to show $m_{\text{sing}} = 0$. Essentially, this property follows from the embedding $W^{1,1} \subset L^q$ for sufficiently small $q > 1$. The details are technical, we therefore present the proof of $m_{\text{sing}} = 0$ in the appendix, see Lemma A.1. Let us emphasize that the proof cannot be trivial: the statement is wrong if no connectedness properties of Ω_ε are imposed. $\square$

Lemma 4.2 (Limit of an integral that is independent of v^ε). *For a smooth test function φ and p of class C^2 in a neighborhood of Γ_0 , there holds, as $\varepsilon \rightarrow 0$,*

$$-\int_{\partial\Omega_\varepsilon} \frac{1}{\varepsilon} n \cdot \nabla p \varphi \rightarrow |\Sigma| \int_{\Gamma_0} [\partial_\nu^2 p \varphi + \partial_\nu p \partial_\nu \varphi] . \quad (4.2)$$

Proof. The full proof is contained in [11], see (25) in Section 4; there, the calculation is performed for $d = 3$, but the dimension is not relevant in the proof. The essential part is the following calculation, which uses that the boundary of the inclusions Σ_ε is the union of the single inclusion boundaries, $\partial\Sigma_\varepsilon = \bigcup_{k \in I_\varepsilon} \partial\Sigma_k^\varepsilon$:

$$\begin{aligned} -\int_{\partial\Omega_\varepsilon} \frac{1}{\varepsilon} n \cdot \nabla p \varphi &= -\sum_{k \in I_\varepsilon} \int_{\partial\Sigma_k^\varepsilon} \frac{1}{\varepsilon} n \cdot (\nabla p \varphi) = \sum_{k \in I_\varepsilon} \int_{\Sigma_k^\varepsilon} \frac{1}{\varepsilon} \nabla \cdot (\nabla p \varphi) \\ &= \sum_{k \in I_\varepsilon} \varepsilon^{-1} |\Sigma| \varepsilon^d \oint_{\Sigma_k^\varepsilon} [\Delta p \varphi + \nabla p \cdot \nabla \varphi] \rightarrow |\Sigma| \int_{\Gamma_0} [\Delta p \varphi + \nabla p \cdot \nabla \varphi] . \end{aligned}$$

An integration by parts on Γ_0 implies (4.2). $\square$

4.1 The limit system

Proposition 4.3 (Equations for weak limits). *Let v^ε be a sequence of solutions to (3.1) satisfying (1.9) and (3.7), let v and μ be limits as in Lemma 4.1, p of class C^2 in a neighborhood of Γ_0 . Then v satisfies the system (1.10). The limit measure μ is given by*

$$\mu = -G \nabla p \mathcal{H}^{d-1}|_{\Gamma_0} . \quad (4.3)$$

Proof. In all subsequent calculations, $\varphi \in C_c^\infty(\Omega)$ is an arbitrary test function.

Step 1: Limits in an integration by parts formula. For an index $j < d$, we use the identity

$$\int_{\Omega_\varepsilon} \partial_j v^\varepsilon \varphi + \int_{\Omega_\varepsilon} v^\varepsilon \partial_j \varphi = e_j \cdot \int_{\partial\Omega_\varepsilon} n v^\varepsilon \varphi .$$

In the limit $\varepsilon \rightarrow 0$, the two integrals on the left hand side converge by (4.1). The limit of the right hand side was determined in (3.11). We find

$$\int_{\Omega} \partial_j v \varphi + \int_{\Omega} e_j \varphi \cdot d\mu + \int_{\Omega} v \partial_j \varphi = - \int_{\Gamma_0} e_j \cdot G \nabla p \varphi . \quad (4.4)$$

An integration by parts is possible in this limit equation, the first and the third integral cancel. The remaining terms provide

$$e_j \cdot \mu = -e_j \cdot G \nabla p \mathcal{H}^{d-1}|_{\Gamma_0}, \quad (4.5)$$

which is the tangential part of the claim (4.3).

Step 2: Limits in the weak form of the equation. We take the limit $\varepsilon \rightarrow 0$ in the original equation (3.2):

$$\begin{aligned} & \int_{\Omega \setminus \Gamma_0} \nabla v \cdot \nabla \varphi + \int_{\Omega} \nabla \varphi \cdot d\mu \xrightarrow{(4.1)} \int_{\Omega_\varepsilon} \nabla v^\varepsilon \cdot \nabla \varphi \\ &= - \int_{\partial\Omega_\varepsilon} \frac{1}{\varepsilon} n \cdot \nabla p \varphi + \int_{\Omega_\varepsilon} \omega^2 v^\varepsilon \varphi \xrightarrow{(4.2)} |\Sigma| \int_{\Gamma_0} (\partial_\nu^2 p \varphi + \partial_\nu p \partial_\nu \varphi) + \int_{\Omega} \omega^2 v \varphi. \end{aligned}$$

If we consider functions φ that are supported on $\Omega \setminus \Gamma_0$, we obtain $-\Delta v = \omega^2 v$ in $\Omega \setminus \Gamma_0$. Having established this fact, we can also insert arbitrary functions $\varphi \in C_c^\infty(\Omega)$ that vanish on Γ_0 and obtain, since $\partial_\nu \varphi$ on Γ_0 can be chosen arbitrarily,

$$e_d \cdot \mu = |\Sigma| \partial_\nu p \mathcal{H}^{d-1}|_{\Gamma_0}. \quad (4.6)$$

In view of (2.8), this is the part of (4.3) that was not verified in Step 1.

We will exploit the above identity of limits further. We can now cancel the factors of $\partial_\nu \varphi$ in Γ_0 -integrals. In the second equality we insert (4.5) and integrate by parts on Γ_0 , in the last equality, we exploit once more (2.8):

$$\begin{aligned} & \int_{\Omega} \nabla v \cdot \nabla \varphi - \int_{\Omega} \omega^2 v \varphi = - \int_{\Omega} \sum_{j=1}^{d-1} \partial_j \varphi d\mu_j + |\Sigma| \int_{\Gamma_0} \partial_\nu^2 p \varphi \\ &= \int_{\Gamma_0} \varphi \left\{ - \sum_{j=1}^{d-1} \partial_j (e_j \cdot G \nabla p) + |\Sigma| \partial_\nu^2 p \right\} = - \int_{\Gamma_0} \varphi \nabla \cdot G \nabla p. \end{aligned}$$

As we have made precise in (1.11), this relation encodes the jump condition

$$[\partial_\nu v] = \nabla \cdot G \nabla p, \quad (4.7)$$

which is the third equation in the limit system (1.10).

Step 3: The jump of values. We start again from an integration by parts formula as in Step 1. Now, we consider the index $j = d$ and $\nu = e_d$ to find

$$\begin{aligned} & \int_{\Omega} v \partial_\nu \varphi + \int_{\Omega} \partial_\nu v \varphi + \int_{\Omega} \varphi d\mu_d \xrightarrow{(4.1)} \int_{\Omega_\varepsilon} v^\varepsilon \partial_\nu \varphi + \int_{\Omega_\varepsilon} \partial_\nu v^\varepsilon \varphi \\ &= e_d \cdot \int_{\partial\Omega_\varepsilon} n v^\varepsilon \varphi \xrightarrow{(3.12)} \int_{\Gamma_0} |\Sigma| \partial_\nu p \varphi - \int_{\Gamma_0} J \cdot \nabla p \varphi. \end{aligned}$$

Relation (4.6) shows that the integral containing μ cancels with the first integral on the right hand side. We obtain, with the jump $[v]$ defined in the sense of traces,

$$\int_{\Gamma_0} [v] \varphi = \int_{\Gamma_0} J \cdot \nabla p \varphi.$$

Since φ is arbitrary, this yields $[v] = J \cdot \nabla p$ of (1.10). The limit system is derived. $\square$

4.2 Inclusions with symmetry

In this short subsection we prove Corollary 1.3. We therefore assume that the inclusion shape Σ is symmetric under all reflections R_j , $j \leq d-1$.

We fix a direction $j \leq d-1$ and study the reflection $R_j : (x_1, \dots, x_d) \mapsto (x_1, \dots, -x_j, \dots, x_d)$. Let w_ξ be the solution to the cell problem for $\xi = e_j$. By symmetry of the inclusion, the reflected function $\tilde{w} := w \circ R_j$ is a solution to the cell problem with $\tilde{\xi} := R_j \xi = -\xi$. We calculate for $i \neq j$ in (2.6)

$$G_{i,j} = e_i \cdot G \xi = e_i \cdot \int_Z \nabla w = e_i \cdot \int_Z \nabla \tilde{w} = e_i \cdot G \tilde{\xi} = -e_i \cdot G \xi,$$

and obtain $G_{i,j} = 0$. Similarly, we calculate in (2.7)

$$\begin{aligned} J \cdot \xi &= - \lim_{\zeta \rightarrow \infty} \int_{\{y_d=\zeta\}} w + \lim_{\zeta \rightarrow -\infty} \int_{\{y_d=\zeta\}} w \\ &= - \lim_{\zeta \rightarrow \infty} \int_{\{y_d=\zeta\}} \tilde{w} + \lim_{\zeta \rightarrow -\infty} \int_{\{y_d=\zeta\}} \tilde{w} = J \cdot \tilde{\xi} = -J \cdot \xi, \end{aligned}$$

and obtain $J \cdot e_j = 0$. The two results imply that G is a diagonal matrix and that J_τ vanishes. Corollary 1.3 is shown.

5 A priori bounds for v^ε

In this section, we study solutions v^ε to System (3.1) with regard to a priori bounds. Under appropriate assumptions, we derive the a priori estimate (1.9).

The derivation of the a priori estimate is based on an explicit construction using solutions of cell problems on $Z := Y \setminus \Sigma$ of (2.1). We consider, for $j \in \{1, \dots, d\}$, the cell solutions w_j of Section 2, solving

$$\Delta w_j = 0 \text{ in } Z, \quad \partial_n w_j = n_j \text{ on } \partial \Sigma. \quad (5.1)$$

For $i, j \in \{1, \dots, d\}$ we use additionally the higher order cell solutions $\psi_{i,j}$ solving

$$\Delta \psi_{i,j} = -2\partial_i w_j \text{ in } Z, \quad \partial_n \psi_{i,j} = \frac{-2}{|\partial \Sigma|} G_{i,j} \text{ on } \partial \Sigma, \quad (5.2)$$

where $G_{i,j} = \int_Z \partial_i w_j$ is as in Section 2. The existence theory for $\psi_{i,j}$ can be performed along the same lines as that for w_j . For the existence of solutions, one has to exploit the integrability condition $\int_Z \Delta \psi_{i,j} = \int_Z (-2\partial_i w_j) = -2G_{i,j} = \int_{\partial \Sigma} \partial_n \psi_{i,j}$.

Our methods require additional regularity assumptions on the inclusion shape Σ . We demand that the solutions of the cell problems (5.1) and (5.2) have the regularity

$$w_j, \psi_{i,j} \in H^2(Z), \quad w_j, \nabla w_j, \psi_{i,j}, \nabla \psi_{i,j} \in L^\infty(Z). \quad (5.3)$$

This is the case, e.g., if $\partial \Sigma$ is of the regularity class C^2 . We note that the solution w_i satisfies additionally the $L^1(Z)$ -estimate

$$\nabla w_j \in L^1(Z). \quad (5.4)$$

This estimate can be concluded by expanding, for arbitrary $j \leq d$, the function $(-\frac{1}{2}, \frac{1}{2})_{\text{per}}^{d-1} \ni \tilde{y} \mapsto \partial_j w_i(\tilde{y}, y_d)$ in a Fourier-series. Averages of this function vanish as was observed in the proof of Lemma 2.2. Since w_i is harmonic, all Fourier coefficients are exponentially decaying in y_d . The exponential decay provides (5.4).

Proposition 5.1 (Construction of $W^{1,1}$ -bounded sequences). *Let $R := (-1, 1)^{d-1} \times (-h, h)$ be a cuboid with height $2h > 0$ and let $g \in C^2(\bar{R}) \cap H^3(R)$ be a function that prescribes boundary data. Let $\Sigma \subset Y$ satisfy the regularity property demanded in (5.3). For a sequence $1/\mathbb{N} \ni \varepsilon \rightarrow 0$, we consider the perforated domains $R_\varepsilon := R \setminus \Sigma_\varepsilon$.*

There exists a sequence $u_\varepsilon : R_\varepsilon \rightarrow \mathbb{R}$ of class $H^2(R_\varepsilon)$ such that the three functions

$$u_\varepsilon \in L^2(R_\varepsilon) \cap W^{1,1}(R_\varepsilon), \quad (5.5)$$

$$\sigma_\varepsilon := \left(\partial_n u_\varepsilon - \frac{1}{\varepsilon} g \cdot n \right) \Big|_{\partial \Sigma_\varepsilon} \in L^\infty(\partial \Sigma_\varepsilon), \quad (5.6)$$

$$\rho_\varepsilon := \Delta u_\varepsilon \in L^\infty(R_\varepsilon), \quad (5.7)$$

are bounded in the indicated function spaces. The functions u_ε can be chosen 1-periodic in the directions e_j , $j = 1, \dots, d-1$, if g has this periodicity property.

Proof. We can provide u_ε with an explicit formula. Using Einstein's summation convention (summing over repeated indices) we set

$$u_\varepsilon(x) := w_j(x/\varepsilon)g_j(x) + \varepsilon\psi_{i,j}(x/\varepsilon)\partial_i g_j(x). \quad (5.8)$$

We note that, by the boundedness (5.3) of w_j and $\psi_{i,j}$, the sequence u_ε is uniformly bounded, hence also bounded in L^2 and in L^1 .

Step 1: $W^{1,1}$ -bound. We start with the highest order term in the gradient of u_ε and calculate its L^1 -norm. We use rescaled cylinders $Z_k^\varepsilon = \varepsilon(k + Z)$, $k \in I_\varepsilon$, where I_ε is the index set corresponding to the domain R . For fixed $j \leq d$ we find

$$\begin{aligned} \int_{R_\varepsilon} \frac{1}{\varepsilon} |\nabla_y w_j(x/\varepsilon)g_j(x)| dx &\leq C \int_{R_\varepsilon} \frac{1}{\varepsilon} |\nabla_y w_j(x/\varepsilon)| dx \\ &\leq C \sum_{k \in I_\varepsilon} \int_{Z_k^\varepsilon} \frac{1}{\varepsilon} |\nabla_y w_j(x/\varepsilon)| dx = C \sum_{k \in I_\varepsilon} \varepsilon^{d-1} \int_Z |\nabla_y w_j(y)| dy \leq C_0 \end{aligned}$$

by (5.4).

The terms $|w_j(x/\varepsilon)\nabla g_j(x)|$, $|\nabla_y \psi_{i,j}(x/\varepsilon)\partial_i g_j(x)|$, and $|\varepsilon\psi_{i,j}(x/\varepsilon)\nabla \partial_i g_j(x)|$ are bounded by the uniform bounds in (5.3). This shows (5.5).

Step 2: Residuals. In order to show (5.6) we calculate for $x \in \partial \Sigma_\varepsilon$, exploiting $n_l \partial_l w_j = n_j$,

$$\begin{aligned} \sigma_\varepsilon(x) &= \partial_n u_\varepsilon(x) - \frac{1}{\varepsilon} g(x) \cdot n(x) \\ &= \frac{1}{\varepsilon} n_l \partial_l w_j(x/\varepsilon)g_j(x) + w_j(x/\varepsilon)n_l \partial_l g_j(x) + n_l \partial_l \psi_{i,j}(x/\varepsilon)\partial_i g_j(x) \\ &\quad + \varepsilon\psi_{i,j}(x/\varepsilon)n_l \partial_l \partial_i g_j(x) - \frac{1}{\varepsilon} n_j g_j(x) \\ &= w_j(x/\varepsilon)n_l \partial_l g_j(x) + n_l \partial_l \psi_{i,j}(x/\varepsilon)\partial_i g_j(x) + \varepsilon\psi_{i,j}(x/\varepsilon)n_l \partial_l \partial_i g_j(x), \end{aligned}$$

which is a bounded sequence of functions.

Similarly, we show (5.7) with a direct calculation, using $\Delta w_j = 0$ and $\Delta \psi_{i,j} = -2\partial_i w_j$ in the last equality:

$$\begin{aligned} \rho_\varepsilon(x) &= \Delta u_\varepsilon \\ &= \frac{1}{\varepsilon^2} \Delta w_j(x/\varepsilon) g_j(x) + \frac{2}{\varepsilon} \partial_i w_j(x/\varepsilon) \partial_i g_j(x) + w_j(x/\varepsilon) \Delta g_j(x) \\ &\quad + \frac{1}{\varepsilon} \Delta \psi_{i,j}(x/\varepsilon) \partial_i g_j(x) + 2\partial_l \psi_{i,j}(x/\varepsilon) \partial_l \partial_i g_j(x) + \varepsilon \psi_{i,j}(x/\varepsilon) \Delta \partial_i g_j(x) \\ &= w_j(x/\varepsilon) \Delta g_j(x) + 2\partial_l \psi_{i,j}(x/\varepsilon) \partial_l \partial_i g_j(x) + \varepsilon \psi_{i,j}(x/\varepsilon) \Delta \partial_i g_j(x). \end{aligned}$$

With (5.3) we obtain that ρ_ε is a bounded sequence. $\square$

The following lemma is related to spectral properties of the Neumann sieve domain: If ω^2 is not an eigenvalue of the negative Laplace operator on Ω , then also the Neumann-sieve problem has no eigenvalues in the vicinity of ω^2 . We show the following boundedness statement, which is used in the derivation of uniform bounds for the corrector.

Lemma 5.2. *Let Ω be a bounded Lipschitz domain, we impose homogeneous and/or periodicity conditions on the boundary. Let $\omega > 0$ be a number with $\omega^2 \notin \sigma(-\Delta)$. Let Ω_ε be the sequence of perforated domains as in (1.5). Then there exists a constant C such that every pair $w_\varepsilon, f_\varepsilon : \Omega_\varepsilon \rightarrow \mathbb{R}$ solving*

$$-\Delta w_\varepsilon = \omega^2 w_\varepsilon + f_\varepsilon \quad (5.9)$$

satisfies

$$\|w_\varepsilon\|_{H^1(\Omega_\varepsilon)} \leq C \|f_\varepsilon\|_{L^2(\Omega_\varepsilon)}. \quad (5.10)$$

Sketch of proof. The proof uses a contradiction argument and the convergence statement of Theorem 1.1 (convergence to a solution without interface). Let us assume that (5.10) does not hold for any C . Exploiting Poincaré's inequality, we find a sequence of pairs $w_\varepsilon, f_\varepsilon : \Omega_\varepsilon \rightarrow \mathbb{R}$ solving (5.9) and satisfying

$$\int_{\Omega_\varepsilon} |\nabla w_\varepsilon|^2 = 1 \quad \forall \varepsilon, \quad \|f_\varepsilon\|_{L^2(\Omega_\varepsilon)} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0. \quad (5.11)$$

Testing (5.9) with w_ε , we find

$$\int_{\Omega_\varepsilon} |\nabla w_\varepsilon|^2 = \omega^2 \int_{\Omega_\varepsilon} |w_\varepsilon|^2 + \int_{\Omega_\varepsilon} w_\varepsilon f_\varepsilon. \quad (5.12)$$

This implies $\|w_\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 \rightarrow 1/\omega^2 \neq 0$.

The convergence of Theorem 1.1 (which is valid also for a sequence $f_\varepsilon \rightarrow 0$ and for other boundary conditions) provides the convergence $\mathcal{P}_\varepsilon w_\varepsilon \rightharpoonup w$, where w is the unique weak solution $w \in H_0^1(\Omega)$ of the limiting Helmholtz problem $-\Delta w = \omega^2 w$. By the eigenvalue assumption, there holds $w = 0$. With the help of Rellich's compactness (using H^1 -bounded extensions of w_ε across the perforation), we obtain that $\mathcal{P}_\varepsilon w_\varepsilon$ converges strongly in $L^2(\Omega)$ to $w = 0$. This is in contradiction with $\|w_\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 \not\rightarrow 0$. $\square$

Theorem 5.3 (Uniform bounds in a periodic problem). *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain with the property $\Omega \cap (\mathbb{R}^{d-1} \times (-h, h)) = (-1, 1)^{d-1} \times (-h, h)$. We consider periodicity boundary conditions on the corresponding lateral parts of $(-1, 1)^{d-1} \times (-h, h)$, any homogeneous boundary conditions on the rest of $\partial\Omega$.*

Let $1/\mathbb{N} \ni \varepsilon \rightarrow 0$ be a sequence, Σ an inclusion with the regularity properties (5.3) and (5.4), and Ω_ε be the perforated domain as in (1.5). For $f \in L^2(\Omega)$ let p^ε be a sequence of solutions to the original problem (1.1), $p^\varepsilon \rightarrow p$ for $p \in C^3(\bar{\Omega}) \cap H^4(\Omega)$. Then the corrector v^ε defined by (1.3) satisfies the $W^{1,1}$ -bound (1.9).

Proof. We set $g(x) := -\nabla p(x)$ and recall that v^ε is characterized by system (3.1). We choose u_ε as in Proposition 5.1 in a strip around the perforation Σ_ε . After a multiplication with a cutoff function we may assume that u_ε is defined in all of Ω .

We introduce a further auxiliary function $\varphi_\varepsilon \in H^1(\Omega_\varepsilon)$ by solving $-\Delta\varphi_\varepsilon = 0$ in Ω_ε with $\partial_n\varphi_\varepsilon = -\sigma_\varepsilon$ on $\partial\Sigma_\varepsilon$, periodicity boundary conditions on the lateral parts of $(-1, 1)^{d-1} \times (-h, h)$. The sequence φ_ε is uniformly bounded in $H^1(\Omega_\varepsilon)$ as can be seen by testing the equation with φ_ε and exploiting a trace theorem.

The difference $\psi_\varepsilon := v^\varepsilon - u_\varepsilon - \varphi_\varepsilon$ satisfies

$$\begin{aligned} -\Delta\psi_\varepsilon - \omega^2\psi_\varepsilon &= \rho_\varepsilon + \omega^2(u_\varepsilon + \varphi_\varepsilon) && \text{in } \Omega_\varepsilon, \\ \partial_n\psi_\varepsilon &= 0 && \text{on } \partial\Sigma_\varepsilon, \end{aligned} \tag{5.13}$$

and periodicity boundary conditions. We invoke Lemma 5.2, which yields the boundedness of ψ_ε in $H^1(\Omega_\varepsilon)$. We conclude that $v^\varepsilon = u_\varepsilon + \psi_\varepsilon + \varphi_\varepsilon$ is a sum of functions that are bounded in $W^{1,1}(\Omega_\varepsilon)$; this shows the assertion. $\square$

Theorem 5.4 (Uniform bounds in a Dirichlet problem). *Let $\Omega = (0, 1)^{d-1} \times (-h, h)$ be a cuboid with height $2h > 0$, let $\Sigma, \varepsilon \rightarrow 0, \Omega_\varepsilon, f, p^\varepsilon$, and p be as in Theorem 5.3. We consider homogeneous Dirichlet boundary conditions on $\partial\Omega$.*

Let $\Sigma \subset (-\frac{1}{2}, \frac{1}{2})^{d-1} \times \mathbb{R}$ possess reflection symmetry in every direction $e_j, j = 1, \dots, d-1$. Then the corrector v^ε defined by (1.3) satisfies the $W^{1,1}$ -bound (1.9).

Proof. We define an extended cuboid by setting $\tilde{\Omega} = (-1, 1)^{d-1} \times (-h, h)$ and extend the functions f and p^ε in an antisymmetric way to all of $\tilde{\Omega}$ by setting $f(x_1, \dots, -x_j, \dots, x_d) = -f(x_1, \dots, x_j, \dots, x_d)$ for $j \in \{1, \dots, d-1\}$, p^ε and v^ε accordingly. The antisymmetric extensions are periodic solutions to the extended problem on $\tilde{\Omega}$. An application of Theorem 5.3 provides the boundedness of v^ε . $\square$

A Bounded sequences in $W^{1,1}(\Omega_\varepsilon)$

We conclude here the proof of Lemma 4.1 by showing the following statement.

Lemma A.1 (Limits of $W^{1,1}(\Omega_\varepsilon)$ -bounded sequences are functions). *Let Ω_ε be as in Section 1.1, see (1.5). For some sequence $\varepsilon \rightarrow 0$, let $(v^\varepsilon)_\varepsilon$ be bounded in $W^{1,1}(\Omega_\varepsilon)$. For a measure $m \in \mathcal{M}(\Omega)$ we assume the convergence of trivial extensions: $\mathcal{P}_\varepsilon v^\varepsilon \xrightarrow{*} m$ in the weak-* sense of measures on Ω . Then m is given by a function, $m = v d\mathcal{L}^d$ with $v \in L^1(\Omega)$. Furthermore, $v \in L^q_{\text{loc}}(\Omega)$ and $\mathcal{P}_\varepsilon v^\varepsilon \rightarrow v$ in $L^1_{\text{loc}}(\Omega)$.*

The proof is inspired by the following observation: Let us assume that the uniform Sobolev embedding estimate $\|u\|_{L^q(\Omega_\varepsilon)} \leq C\|u\|_{W^{1,1}(\Omega_\varepsilon)}$ holds on Ω_ε . Under this assumption, the sequence $\mathcal{P}_\varepsilon v^\varepsilon$ is bounded in $L^q(\Omega)$; it therefore possesses, along a subsequence, a weak limit $v \in L^q(\Omega)$. Compactness of the embedding can imply additionally the strong convergence.

We emphasize that the Sobolev embedding estimate does not hold when the obstacle set Σ has a complement with a bounded connected component. Also the statement of the lemma is wrong in this case, since m can have a singular part which is supported on Γ_0 .

Proof. We choose $q > 1$ with $1 - \frac{d}{1} > -\frac{d}{q}$ or, equivalently, $q < d/(d-1)$. We recall the inclusion property $\Sigma \subset I^d = (-\frac{1}{2}, \frac{1}{2})^d$ and the fact that the complement $I^d \setminus \Sigma$ is connected.

The limit measure m can be decomposed into an absolute continuous part and a singular part (with respect to the Lebesgue measure $\mathcal{L}^d$), we write $m = v d\mathcal{L}^d + m_{\text{sing}}$ with $v \in L^1(\Omega)$. One of our aims is to show $m_{\text{sing}} = 0$.

Step 1: Estimate in a strip near the perforation. In this step we want to estimate the L^1 -norm of v^ε in the vicinity of the perforation. We select arbitrary subsets $\tilde{\Gamma} \subset \hat{\Gamma} \subset \Gamma$, each subset open and compactly contained in the next larger set. We define a thin strip as $S_\varepsilon^+ := \hat{\Gamma} \times (\varepsilon, 2\varepsilon) \subset \Omega_\varepsilon$. For $h > 0$ sufficiently small, we introduce a cylindrical domain of order one, overlapping with the strip: $Q_\varepsilon^+ := \hat{\Gamma} \times (\varepsilon, h+\varepsilon) \subset \Omega_\varepsilon$. In the domain Q_ε^+ , the Sobolev embedding results can be used since, up to a vertical shift of length ε , the domains Q_ε^+ are independent of ε . The Sobolev embedding implies boundedness of $v^\varepsilon \in L^q(Q_\varepsilon^+)$. In particular, we find for the strip S_ε^+ and the dual exponent $p = q/(q-1)$ with Hölder's inequality the bound

$$\int_{S_\varepsilon^+} |v^\varepsilon| \leq \left(\int_{S_\varepsilon^+} |v^\varepsilon|^q \right)^{1/q} |S_\varepsilon^+|^{1/p} \leq C\varepsilon^{(q-1)/q}. \quad (\text{A.1})$$

Step 2: Poincaré estimate. Let $Y = (-\frac{1}{2}, \frac{1}{2})_{\text{per}}^{d-1} \times \mathbb{R}$ and $Z = Y \setminus \Sigma$ be defined as in the cell-problem, see (2.1). We furthermore set $Z^+ := Z \cap \{y_d \in (1, 2)\}$ and $Z^0 := Z \cap \{y_d \in (-1, 1)\}$. By connectedness of Z , for some constant C_P , there holds the Poincaré estimate

$$\int_{Z^0} |u| \leq C_P \left\{ \int_{Z^+} |u| + \int_{Z^+ \cup Z^0} |\nabla u| \right\} \quad (\text{A.2})$$

for every $u \in W_{\text{loc}}^{1,1}(Z)$.

The estimate (A.2) can be scaled to domains $Z_k^\varepsilon := \varepsilon(k + Z)$, $Z_k^{0,\varepsilon} := \varepsilon(k + Z^0)$, $Z_k^{+,\varepsilon} := \varepsilon(k + Z^+)$. We sum the scaled estimate over all $k \in \mathbb{Z}^{d-1}$ with $(\tilde{\Gamma} \times \{0\}) \cap Z_k^\varepsilon \neq \emptyset$ and obtain, for $\varepsilon > 0$ sufficiently small,

$$\begin{aligned} \int_{(\tilde{\Gamma} \times (-\varepsilon, \varepsilon)) \cap \Omega_\varepsilon} |v^\varepsilon| &\leq \sum_k \int_{Z_k^{0,\varepsilon}} |v^\varepsilon| \leq C_P \sum_k \left\{ \int_{Z_k^{+,\varepsilon}} |v^\varepsilon| + \varepsilon \int_{Z_k^\varepsilon \cap \{|x_d| < 2\varepsilon\}} |\nabla v^\varepsilon| \right\} \\ &\leq C_P \int_{S_\varepsilon^+} |v^\varepsilon| + C_P \varepsilon \|\nabla v^\varepsilon\|_{L^1(\Omega_\varepsilon)}. \end{aligned}$$

Combining this with estimate (A.1) of Step 1 we find

$$\int_{(\tilde{\Gamma} \times (-\varepsilon, \varepsilon)) \cap \Omega_\varepsilon} |v^\varepsilon| \rightarrow 0 \quad (\text{A.3})$$

as $\varepsilon \rightarrow 0$. This verifies that the contributions of v^ε in an ε -strip around the perforations is small (measured in L^1).

Step 3: Conclusion. Let $\tilde{\Omega} \subset \Omega$ be a compactly contained subset. We cover the set $\tilde{\Omega}$ with four sets: The strip $S_\varepsilon^0 := (\tilde{\Gamma} \times (-\varepsilon, \varepsilon))$ for a sufficiently large subset $\tilde{\Gamma} \subset \Gamma$, the two sets $Q_\varepsilon^\pm := \hat{\Gamma} \times (\pm h \pm \varepsilon, \pm \varepsilon)$, and an ε -independent subset $Q_0 \subset \Omega_\varepsilon$. We consider the truncated variable $\tilde{v}^\varepsilon : \Omega \rightarrow \mathbb{R}$, defined as $\tilde{v}^\varepsilon(x) := 0$ for $x \in S_\varepsilon^0$ and $\tilde{v}^\varepsilon(x) = v^\varepsilon(x)$ else.

As observed in Step 1, the function $\tilde{v}^\varepsilon$ is bounded in $L^q(Q_\varepsilon^\pm)$ and in $L^q(Q)$ by the Sobolev embedding. Since $\tilde{v}^\varepsilon$ vanishes in the strip S_ε^0 , the sequence $\tilde{v}^\varepsilon$ is bounded in $L^q(\tilde{\Omega})$. Because of the bound (A.3), the measure-valued limit of $\tilde{v}^\varepsilon$ is identical with that of v^ε , hence $\tilde{v}^\varepsilon \xrightarrow{*} m$. The $L^q(\tilde{\Omega})$ -boundedness of $\tilde{v}^\varepsilon$ implies that the limit measure m is given by an $L^q(\tilde{\Omega})$ -function on $\tilde{\Omega}$. This shows $m_{\text{sing}} = 0$ and $v \in L_{\text{loc}}^q(\Omega)$.

The compactness of the Sobolev embedding also shows $v^\varepsilon \rightarrow v$ in $L^q(Q)$ and in $L^q(Q_\varepsilon^\pm)$. For the latter result we use that ε -translations of v converge in L^q locally to v . Since the strip S_ε^0 has only a small L^1 -contribution, we conclude also $v^\varepsilon \rightarrow v$ in $L^1(\tilde{\Omega})$. $\square$

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