

Lattice dynamics on large time scales and dispersive effective equations

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Abstract: We investigate the long time behavior of waves in crystals. Starting from a linear wave equation on a discrete lattice with periodicity $\varepsilon > 0$, we derive the continuum limit equation for time scales of order ε^{-2} . The effective equation is a weakly dispersive wave equation of fourth order. Initial values with bounded support result in ring-like solutions and we characterize the dispersive long-time behavior of the radial profiles with a linearized KdV equation of third order.

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1 Introduction

Our aim is to describe the oscillations of atoms in a crystal lattice. In the simplest setting of the problem, the continuum limit provides a linear wave equation of second order. Due to its simple structure, this limit model fails to describe certain phenomena that can be observed in crystals: Nonlinear behavior and dispersion. We concentrate here on the latter and justify a limit model that captures dispersive effects.

The discrete model is constructed from a regular rectangular lattice as follows. The lattice points are $\gamma \in \varepsilon\mathbb{Z}^d$ where $\varepsilon > 0$ is the periodicity and $d \geq 1$ the space dimension. Considering only displacements in one fixed direction, the unknown in the point γ at time $t \in [0, \infty)$ is $u^\varepsilon(\gamma, t) \in \mathbb{R}$. The evolution equation relates the acceleration $\partial_t^2 u^\varepsilon(\gamma, t)$ to the displacements $u^\varepsilon(\gamma', t)$ in neighboring points γ' . Restricting ourselves to a linear model we consider

$$\partial_t^2 u^\varepsilon(\gamma, t) = \frac{1}{\varepsilon^2} \sum_{j \in \mathbb{Z}^d} a_j u^\varepsilon(\gamma + \varepsilon j, t) \quad (1.1)$$

for certain prescribed interaction coefficients $(a_j)_{j \in \mathbb{Z}^d}$, $a_j \in \mathbb{R}$ with $\sum_j a_j = 0$. If we prescribe low frequency initial data $u_0^\varepsilon, u_1^\varepsilon : \mathbb{R}^d \rightarrow \mathbb{R}$ (e.g. by choosing the initial data independent of ε), then the continuum limit of (1.1) is the linear second order wave equation

$$\partial_t^2 u = A D_x^2 u. \quad (1.2)$$

Here, the elliptic operator $AD_x^2 = \sum_{i,j=1}^d A_{i,j} \partial_{x_i} \partial_{x_j}$ is given by an x -independent effective tensor $A \in \mathbb{R}^{d \times d}$ which can be calculated from the interaction coefficients $(a_j)_{j \in \mathbb{Z}^d}$. Equation (1.2) is a valid approximation for time intervals $[0, T]$; since the time interval is fixed, the equation fails to capture dispersive effects. Our interest is to derive a dispersive continuum limit equation that is valid on time intervals $[0, T\varepsilon^{-2}]$.

One-dimensional example. We have chosen here a natural scaling of the equation. The simplest example is obtained by considering the one-dimensional case $d = 1$ and the next-neighbor interaction with $a_1 = a_{-1} = 1$, $a_0 = -2$, and $a_j = 0$ for every j with $|j| > 1$. In this case, the right hand side of (1.1) is the discrete Laplacian of $u^\varepsilon(\cdot, t)$ in the point γ . The homogenized equation is (1.2) with $A = 1$.

We consider two alternative approaches to characterize the evolution over time-scales of order ε^{-2} :

- A) Derivation of continuum equations with ε -dependent coefficients. This is done in Section 3. The advantage is that the effective equations have constant coefficients. The disadvantage is that the equations have to be solved on large time intervals.
- B) Effective equations with ε -independent coefficients (profile equations). Approximations to the solutions of the original lattice equation are obtained via ε -dependent transforms. This program is carried out in Sections 4 and 5.

A) Dispersive continuum limit. One of our results is that the following weakly dispersive equation describes the long time behavior of the discrete system:

$$\partial_t^2 w^\varepsilon = AD_x^2 w^\varepsilon + \varepsilon^2 ED_x^2 \partial_t^2 w^\varepsilon - \varepsilon^2 FD_x^4 w^\varepsilon. \quad (1.3)$$

We note that the limit system is continuous with x -independent coefficients, we may therefore call it a continuum limit of the discrete system. On the other hand, we note that the lattice constant ε is not set to zero, but it appears in the equations. The result of Theorem 3.1 is the following: Let u^ε be a solution of the discrete system (1.1) on a time interval $[0, T/\varepsilon^2]$ and let $w^\varepsilon : \mathbb{R}^d \times [0, T/\varepsilon^2] \rightarrow \mathbb{R}$ be a solution to (1.3) on the same time interval. Then the two solutions differ in the energy norm only in the order of ε . The estimate is relevant since the deviation of both functions from the solution of (1.2) is of order 1.

Our results can be compared to known results for the continuous problem. In the continuous case one starts from the wave equation with the coefficient $a(x/\varepsilon)$ in the elliptic operator. As for discrete systems, dispersive effects are observed [7, 8]. These can be understood with the help of the dispersion relation that is obtained with the help of Bloch expansions [15]. Also in the continuous setting, one can derive the dispersive limit equation (1.3) for large time scales [4, 5, 11].

In (1.3), the effective tensors $E \in \mathbb{R}^{d \times d}$ and $F \in \mathbb{R}^{d \times d \times d \times d}$ are symmetric and positive semi-definite, we use the notation $FD_x^4 = \sum_{i,j,k,l=1}^d F_{i,j,k,l} \partial_{x_i} \partial_{x_j} \partial_{x_k} \partial_{x_l}$. The x -independent coefficient tensors are obtained from a Taylor expansion of the dispersion relation. In the one-dimensional case of [11], A , E and F are positive real numbers. The quality of the effective equation (1.3) is also studied numerically, see [1, 2, 4, 5] for numerical studies and generalizations.

In the present work, we obtain a similar approximation result for the discrete wave equation (1.1). The approach is similar in the sense that we start from a

representation formula, which is analyzed in the limit $\varepsilon \rightarrow 0$. In the discrete setting, the representation with Fourier transforms is possible, we hence do not have to use Bloch transforms. This simplifies the proof of the approximation result considerably.

In classical approaches, the dynamics of (1.1) are analyzed in terms of the dispersion relation, see e.g. [6, 13, 14]. In fact, our proofs are also based on an approximation of the dispersion relations of (1.1) and (1.3) to all relevant orders.

B) Profile equations for ring-like solutions. In the second part of this work, starting with Section 4, we characterize the long time behavior of solutions with a family of one-dimensional profile equations. If initial data are given by a fixed $L^2(\mathbb{R}^d)$ function, then the solution at time $t = \tau/\varepsilon^2$ is large only in a neighborhood of a sphere with radius $r = ct = c\tau/\varepsilon^2$, where $c > 0$ is the effective wave velocity. We describe the profile of the wave in this neighborhood in dimension $d = 1$ and in dimension $d = 2$. Our results imply, in particular, that the profiles satisfy the linearized KdV equations

$$\partial_\tau V^\varepsilon(z, \tau; q) = b(q) \partial_z^3 V^\varepsilon(z, \tau; q). \quad (1.4)$$

In this equation, $z \in \mathbb{R}$ is the radial parameter (the radial distance to the sphere $|x| = ct$), $\tau = \varepsilon^2 t$ is the rescaled time and $q \in S^{d-1}$ is the direction of propagation. The effective coefficient $b(q) \in \mathbb{R}$ is obtained from the Taylor expansion of the dispersion relation of (1.1). The initial data for (1.4) are extracted from the initial data u_0 and u_1 ; the construction is dimension dependent.

Our results are quite strong as they provide approximate solutions v^ε . The functions v^ε are easy to calculate and they approximate the lattice solution u^ε to (1.1). The construction of v^ε starts from the initial data u_0^ε and defines initial data V_0^ε for (1.4) from u_0^ε ; the solutions of (1.4) are used as profiles to construct v^ε as a ring type solution. Our results in Theorem 4.1 and Theorem 5.3 establish that v^ε is an approximation u^ε on time intervals $(0, T/\varepsilon^2)$.

The fact that a linearized KdV equation should describe the profile of a ring-like solution is well-known. In the one-dimensional case, it is possible to study a solution to (1.3) in a frame of coordinates that moves with the effective speed $\sqrt{A}$. Formal calculations show that the shifted solution solves, approximately, the linearized KdV equation (1.4). The observation was made rigorous in [12], Theorem 2.7, where the error was shown to vanish in the limit $\varepsilon \rightarrow 0$. In space dimension $d = 2$, a result was established in Proposition 3.1 of [4]: If the wave profile of a solution w^ε to (1.3) converges in the sense of distributions, then the limiting profile satisfies a linearized KdV equation. We remark that, in [4], the height of the wave profile was not scaled with the factor $|ct|^{-1/2}$ that we use in (4.3); without this factor, one obtains the “cylindrical” KdV equation. Our result has the advantage that the initial data can be extracted explicitly from the initial data of the lattice equation.

Let us compare our result also with [9], where the dynamics of the nonlinear FPU-lattice are studied. For the nonlinear dynamics, a nonlinear KdV equation is obtained in [9], but we note that only solutions to profile equations can be compared in this case. Since we deal only with the linear case, our result can be more general: For fixed initial data in space dimension $d = 1$ or $d = 2$, the solution is a ring wave with a profile described by (1.4).

2 Preliminaries

We study (1.1) with initial conditions

$$u^\varepsilon(\gamma, 0) = u_0^\varepsilon(\gamma), \quad \partial_t u^\varepsilon(\gamma, 0) = u_1^\varepsilon(\gamma) \quad \forall \gamma \in \varepsilon\mathbb{Z}^d, \quad (2.1)$$

where $u_0^\varepsilon, u_1^\varepsilon : \mathbb{R}^d \rightarrow \mathbb{R}$ are given functions on all of $\mathbb{R}^d$.

Fourier transformation

We define the Fourier transform for functions on the lattice by setting, for $k \in \mathbb{R}^d$,

$$\hat{u}^\varepsilon(k, t) := \begin{cases} \varepsilon^d \sum_{\gamma \in \varepsilon\mathbb{Z}^d} e^{-ik \cdot \gamma} u^\varepsilon(\gamma, t) & \text{for } k \in \varepsilon^{-1}(-\pi, \pi)^d, \\ 0 & \text{else.} \end{cases} \quad (2.2)$$

The Fourier transform $\hat{u}^\varepsilon$ of the solution u^ε satisfies, for every $k \in \mathbb{R}^d$:

$$\begin{aligned} \partial_t^2 \hat{u}^\varepsilon(k, t) &= \varepsilon^d \sum_{\gamma \in \varepsilon\mathbb{Z}^d} e^{-ik \cdot \gamma} \frac{1}{\varepsilon^2} \sum_{j \in \mathbb{Z}^d} a_j u^\varepsilon(\gamma + \varepsilon j, t) \\ &= \varepsilon^d \sum_{j \in \mathbb{Z}^d} \frac{1}{\varepsilon^2} a_j e^{ik \cdot \varepsilon j} \sum_{\gamma \in \varepsilon\mathbb{Z}^d} e^{-ik \cdot \varepsilon j - ik \cdot \gamma} u^\varepsilon(\gamma + \varepsilon j, t) \\ &= \frac{1}{\varepsilon^2} \left[\sum_{j \in \mathbb{Z}^d} a_j e^{ik \cdot \varepsilon j} \right] \varepsilon^d \sum_{\gamma' \in \varepsilon\mathbb{Z}^d} e^{-ik \cdot \gamma'} u^\varepsilon(\gamma', t). \end{aligned}$$

We obtain, introducing the lattice dispersion relation

$$\omega_l(\varepsilon k)^2 := - \sum_{j \in \mathbb{Z}^d} a_j e^{ik \cdot \varepsilon j}, \quad (2.3)$$

the evolution equation for the Fourier transform in the simple form

$$\partial_t^2 \hat{u}^\varepsilon(k, t) = - \frac{1}{\varepsilon^2} \omega_l(\varepsilon k)^2 \hat{u}^\varepsilon(k, t). \quad (2.4)$$

We illustrate the scaling with a calculation for the above one-dimensional example. We find $\omega_l(\varepsilon k)^2 := - \sum_{j \in \mathbb{Z}} a_j e^{ik \cdot \varepsilon j} = -(e^{ik\varepsilon} - 2 + e^{-ik\varepsilon}) = \varepsilon^2 k^2 - \frac{1}{12} \varepsilon^4 k^4 \pm \dots$. To leading order, the dispersion relation is that of an isotropic wave equation, $\omega_l(\varepsilon k)^2 = \varepsilon^2 k^2$.

Diagonalization and explicit solution

The evolution law (2.4) can be diagonalized by introducing the functions

$$\hat{\psi}_\pm^{u, \varepsilon}(k, t) := \frac{1}{\sqrt{2}} \left(\frac{\omega_l(\varepsilon k)}{\varepsilon} \hat{u}^\varepsilon(k, t) \pm i \partial_t \hat{u}^\varepsilon(k, t) \right). \quad (2.5)$$

For the time derivative of these functions we find by (2.4)

$$\partial_t \hat{\psi}_\pm^{u, \varepsilon}(k, t) = \frac{1}{\sqrt{2}} \left(\frac{\omega_l(\varepsilon k)}{\varepsilon} \partial_t \hat{u}^\varepsilon(k, t) \mp i \frac{\omega_l(\varepsilon k)^2}{\varepsilon^2} \hat{u}^\varepsilon(k, t) \right)$$

$$\begin{aligned}
&= -\frac{i\omega_l(\varepsilon k)}{\varepsilon} \frac{1}{\sqrt{2}} \left(i\partial_t \hat{u}^\varepsilon(k, t) \pm \frac{\omega_l(\varepsilon k)}{\varepsilon} \hat{u}^\varepsilon(k, t) \right) \\
&= \mp \frac{i\omega_l(\varepsilon k)}{\varepsilon} \hat{\psi}_\pm^{u, \varepsilon}(k, t).
\end{aligned}$$

This provides the explicit solution formula

$$\hat{\psi}_\pm^{u, \varepsilon}(k, t) = e^{\mp i[\omega_l(\varepsilon k)/\varepsilon]t} \hat{\psi}_\pm^{u, \varepsilon}(k, 0). \quad (2.6)$$

The initial data are given by $\hat{\psi}_\pm^{u, \varepsilon}(k, 0) = \hat{\psi}_{\pm, 0}^{u, \varepsilon}(k)$. These two functions are related to the initial data $\hat{u}_0^\varepsilon$ and $\hat{u}_1^\varepsilon$ via equation (2.5). Formula (2.5) expresses $\hat{\psi}_\pm^{u, \varepsilon}$ in terms of $\hat{u}^\varepsilon$ and $\partial_t \hat{u}^\varepsilon$, but we can also reconstruct with

$$\hat{u}^\varepsilon(k, t) = \frac{\varepsilon}{\omega_l(\varepsilon k)\sqrt{2}} \left(\hat{\psi}_+^{u, \varepsilon}(k, t) + \hat{\psi}_-^{u, \varepsilon}(k, t) \right), \quad (2.7)$$

and with a similar expression for $\partial_t \hat{u}^\varepsilon(k, t)$. The reconstruction formulas allow also to write $\hat{u}_0^\varepsilon$ and $\hat{u}_1^\varepsilon$ in terms of $\hat{\psi}_{\pm, 0}^{u, \varepsilon}(k)$.

What makes the transformation formulas slightly non-trivial is the fact that they involve explicit dependences on ε . For example: If we choose initial data $\hat{\psi}_{\pm, 0}^{u, \varepsilon}$ without ε -dependence, then the initial data for u^ε are ε -dependent. We always assume that the involved functions have compact support in Fourier space (a property that is not affected by the transformation).

The result of this work is a characterization of the dispersive long time behavior of u^ε with partial differential equations. We achieve this goal with the help of the Fourier representation (2.6) of solutions.

With the above Fourier transform and diagonalization procedure we followed [10], see in particular their evolution equation (2.11). The only difference is in the choice of the grid spacing: We have chosen a grid spacing ε in order that we do not have to rescale solutions. In particular, the initial data can be prescribed by ε -independent functions and the dispersive limit equation turns out to be as in [4], where the oscillations of the medium are also on the spatial scale ε .

Expansion of the dispersion relation We have to define the effective tensors. We start from a Taylor expansion of the (lattice) dispersion relation (2.3) in $k = 0$:

$$\omega_l^2(k) = Ak \otimes k + Ck \otimes k \otimes k \otimes k + O(|k|^6) \quad (2.8)$$

We emphasize that the second order tensor $A \in \mathbb{R}^{d \times d}$ and the fourth order tensor $C \in \mathbb{R}^{d \times d \times d \times d}$ are defined by the Taylor expansion (2.8). The assumptions are that ω_l^2 is real and of class C^6 in a neighborhood of $\tilde{k} = 0$, which is satisfied e.g. if the coefficients $(a_j)_j$ have only finitely many non-zero entries and satisfy $a_j = a_{-j}$. We furthermore assume that A is symmetric and positive definite. With the help of A we define a (homogenized) dispersion relation through

$$\omega_0^2(k) := Ak \otimes k. \quad (2.9)$$

We also require approximations for ω_l :

$$\begin{aligned}
\omega_l(k) &= (Ak \otimes k (\text{Id} + (Ak \otimes k)^{-1} Ck \otimes k \otimes k \otimes k) + O(|k|^6))^{\frac{1}{2}} \\
&= \omega_0(k) + b(k) + O(|k|^4) \quad \text{for } |k| \ll 1,
\end{aligned} \quad (2.10)$$

where $b(k) = \frac{1}{2\omega_0(k)} Ck \otimes k \otimes k \otimes k$ is a 3-homogeneous function.

Remark 2.1. 1.) The 3-homogeneous function b is even, $b(k) = b(-k)$.

2.) The function b is in general not a polynomial in k . For example, in the simple case of a two-dimensional square lattice with nearest neighbor interactions, we find $b(k) = -(k_1^4 + k_2^4)/(24\sqrt{k_1^2 + k_2^2})$.

3 Comparison with a weakly dispersive equation

In this section, we compare the solution u^ε of the lattice wave equation (1.1) with the solution w^ε of the weakly dispersive wave equation (1.3). On a fixed time interval $[0, T]$ the two solutions both coincide to leading order in ε with the solution of the homogenized wave equation (1.2). Here, we study time intervals $t \in [0, T/\varepsilon^2]$ on which (1.2) is not a valid approximation to leading order. Our result is that, on the other hand, the two solutions u^ε and w^ε coincide to leading order.

Considering expansion (2.8), it is tempting to compare the solution u^ε with the solution u to the equation $\partial_t^2 u = A D_x^2 u - \varepsilon^2 C D_x^4 u$. This latter equation has been suggested in another context already by Boussinesq and is named after him. In the 1-dimensional case discussed above there holds $\omega_l(\tilde{k})^2 = \tilde{k}^2 - \frac{1}{12}\tilde{k}^4$, hence $A = 1$ and $C = -\frac{1}{12}$; the Boussinesq equation reads $\partial_t^2 u = \partial_x^2 u + \frac{1}{12}\varepsilon^2 \partial_x^4 u$. This equation is called a “bad Boussinesq equation” since it cannot be solved easily: ∂_x^2 is a negative operator and ∂_x^4 is a positive operator. One way to proceed is to replace in the equation $\partial_x^4 u$ by $\partial_t^2 \partial_x^2 u$, which is correct to highest order [3].

We follow this idea of replacing the highest order term (at least in parts) by mixed derivatives. We rely on Lemma 2.5 of [5], which provides the following: Given a symmetric and positive definite $A \in \mathbb{R}^{d \times d}$ and a fourth order tensor $C \in \mathbb{R}^{d \times d \times d \times d}$, there exist symmetric and positive semi-definite tensors $E \in \mathbb{R}^{d \times d}$ and $F \in \mathbb{R}^{d \times d \times d \times d}$ such that

$$C\tilde{k} \otimes \tilde{k} \otimes \tilde{k} \otimes \tilde{k} = -E\tilde{k} \otimes \tilde{k} A\tilde{k} \otimes \tilde{k} + F\tilde{k} \otimes \tilde{k} \otimes \tilde{k} \otimes \tilde{k} \quad \forall \tilde{k} \in \mathbb{R}^d. \quad (3.1)$$

We choose E and F as described. We conclude from (2.8) and (3.1) the approximation property

$$\omega_l^2(\tilde{k}) = A\tilde{k} \otimes \tilde{k} - E\tilde{k} \otimes \tilde{k} \omega_l^2(\tilde{k}) + F\tilde{k} \otimes \tilde{k} \otimes \tilde{k} \otimes \tilde{k} + O(|\tilde{k}|^6) \quad (3.2)$$

as $\tilde{k} \rightarrow 0$. Replacing once more Fourier symbols by derivatives we arrive formally at the weakly dispersive wave equation (1.3).

Comparison of solutions. Let us now change the perspective: We consider the evolution equation (1.3) as given and want to analyze its solutions w^ε . Since solutions can be expanded in Fourier space, they can be characterized by the dispersion relation ω_d of the equation (1.3) (the subscript d recalls that the equation is *dispersive*). We define ω_d implicitly through

$$\omega_d^2(\tilde{k}) = A\tilde{k} \otimes \tilde{k} - E\tilde{k} \otimes \tilde{k} \omega_d^2(\tilde{k}) + F\tilde{k} \otimes \tilde{k} \otimes \tilde{k} \otimes \tilde{k}. \quad (3.3)$$

By property (3.1) of E and F , the function ω_d^2 has the same Taylor expansion to fourth order in 0 as ω_l^2 , i.e. ω_d^2 satisfies (2.8). In particular, the difference satisfies $\omega_d^2(\tilde{k}) - \omega_l^2(\tilde{k}) = O(|\tilde{k}|^6)$ and therefore

$$|\omega_d(\tilde{k}) - \omega_l(\tilde{k})| = O(|\tilde{k}|^4) \quad \text{as } \tilde{k} \rightarrow 0. \quad (3.4)$$

We used here that A is positive definite.

For sufficiently small $\tilde{k} = \varepsilon k$ we can solve relation (3.3) for $\omega_d^2(\varepsilon k)$,

$$\frac{1}{\varepsilon^2} \omega_d^2(\varepsilon k) = \frac{Ak \otimes k + \varepsilon^2 Fk \otimes k \otimes k \otimes k}{1 + \varepsilon^2 Ek \otimes k}. \quad (3.5)$$

After a (continuous) Fourier-transform in x with the dual variable k , the evolution equation (1.3) reads (D_x is replaced by ik):

$$\partial_t^2 \hat{w}^\varepsilon(k, t) = -Ak \otimes k \hat{w}^\varepsilon(k, t) - \varepsilon^2 Ek \otimes k \partial_t^2 \hat{w}^\varepsilon(k, t) - \varepsilon^2 Fk \otimes k \otimes k \otimes k \hat{w}^\varepsilon(k, t). \quad (3.6)$$

If we assume that the initial values have a compact support in Fourier space, then the parameter k takes only values in a bounded set. In this case, the parameter εk is always small and we can solve (3.6) for $\partial_t^2 \hat{w}^\varepsilon(k, t)$, leading to the same expression as in (3.5); more precisely,

$$\partial_t^2 \hat{w}^\varepsilon(k, t) = -\frac{1}{\varepsilon^2} \omega_d^2(\varepsilon k) \hat{w}^\varepsilon(k, t). \quad (3.7)$$

This evolution equation is solved for every k on the time interval $t \in [0, T/\varepsilon^2]$. After a transformation as for u^ε , we define the transformed quantity $\hat{\Psi}_\pm^{w, \varepsilon}$ as in (2.5), i.e.

$$\hat{\psi}_\pm^{w, \varepsilon}(k, t) := \frac{1}{\sqrt{2}} \left(\frac{1}{\varepsilon} \omega_d(\varepsilon k) \hat{w}^\varepsilon(k, t) \pm i \partial_t \hat{w}^\varepsilon(k, t) \right). \quad (3.8)$$

The new unknowns have the simple explicit solution formula

$$\hat{\psi}_\pm^{w, \varepsilon}(k, t) = e^{\mp i[\omega_d(\varepsilon k)/\varepsilon]t} \hat{\psi}_{\pm, 0}^{w, \varepsilon}(k). \quad (3.9)$$

Initial values. The solution of the ε -problem (1.1) is determined by the initial values u_0^ε and u_1^ε . After a Fourier transform (2.5) provides the two initial data $\hat{\psi}_{\pm, 0}^{u, \varepsilon}(k)$ that are used in the solution formula (2.6).

The initial data w_0^ε and w_1^ε of the weakly dispersive equation (1.3) are treated accordingly: Continuous Fourier transform and (2.5) define the initial data $\hat{\psi}_{\pm, 0}^{w, \varepsilon}(k)$. We note that we can also reconstruct w from $\hat{\psi}_\pm^{w, \varepsilon}$ with (2.7), except that we have to use the appropriate dispersion relation: $\hat{w}_0^\varepsilon(k) = \frac{\varepsilon}{\omega_d(\varepsilon k)\sqrt{2}} (\hat{\psi}_{+, 0}^{u, \varepsilon}(k) + \hat{\psi}_{-, 0}^{u, \varepsilon}(k))$. Similarly, we can reconstruct $\hat{w}_1^\varepsilon$.

Theorem 3.1 (Comparison of solutions). *Let initial values be given by sequences $(u_0^\varepsilon, u_1^\varepsilon)$ and $(w_0^\varepsilon, w_1^\varepsilon)$ such that, for $C_0 > 0$,*

$$\|\hat{w}_0^\varepsilon - \hat{u}_0^\varepsilon\|_{L^2(\mathbb{R}^d)} + \|\hat{w}_1^\varepsilon - \hat{u}_1^\varepsilon\|_{L^2(\mathbb{R}^d)} \leq C_0 \varepsilon. \quad (3.10)$$

We recall that the Fourier transform of u_0 and u_1 was set to zero for $k \notin \varepsilon^{-1}(-\pi, \pi)^d$. We assume that, for a fixed bounded domain $S_\psi \subset \mathbb{R}^d$, the Fourier transforms of all initial data are supported in S_ψ and bounded in $L^2(\mathbb{R}^d)$.

On the dispersion relation ω_l from (2.3) of the lattice system we assume that it satisfies (2.8) for tensors A and C , A symmetric and positive definite. Let E and F be symmetric and positive semi-definite tensors that satisfy (3.1). We recall that, in this case, ω_l and ω_d from (3.3) satisfy (3.4).

In this situation we consider the solution $u^\varepsilon(., t)$ of the lattice equation (1.1) and the solution $w^\varepsilon(., t)$ of the weakly dispersive equation (1.3), additionally the

transformed solutions $\hat{\psi}_{\pm}^{w,\varepsilon}(\cdot, t)$ of (3.8) and $\hat{\psi}_{\pm}^{u,\varepsilon}(\cdot, t)$ of (2.5). Then the transformed solutions are comparable on large time intervals for both signs, “+” and “-”:

$$\sup_{t \leq T/\varepsilon^2} \left\| \hat{\psi}_{\pm}^{w,\varepsilon}(\cdot, t) - \hat{\psi}_{\pm}^{u,\varepsilon}(\cdot, t) \right\|_{L^2(\mathbb{R}^d)} \leq C\varepsilon. \quad (3.11)$$

The original solutions are nearby in energy norm: With ω_0 from (2.9) holds

$$\sup_{t \leq T/\varepsilon^2} \left\{ \left\| \frac{\omega_0(\varepsilon \cdot)}{\varepsilon} [\hat{w}^\varepsilon(\cdot, t) - \hat{u}^\varepsilon(\cdot, t)] \right\|_{L^2(\mathbb{R}^d)} + \left\| \partial_t \hat{w}^\varepsilon(\cdot, t) - \partial_t \hat{u}^\varepsilon(\cdot, t) \right\|_{L^2(\mathbb{R}^d)} \right\} \leq C\varepsilon. \quad (3.12)$$

The constant C depend only on ω_l , C_0 , S_ψ , and T .

Proof. For the proof of (3.11) it suffices to insert the explicit solution formulas (2.6) and (3.9). The calculation for “+” is

$$\begin{aligned} & \sup_{t \leq T/\varepsilon^2} \left\| \hat{\psi}_+^{w,\varepsilon}(\cdot, t) - \hat{\psi}_+^{u,\varepsilon}(\cdot, t) \right\|_{L^2(\mathbb{R}^d)}^2 \\ &= \sup_{t \leq T/\varepsilon^2} \int_{\mathbb{R}^d} \left| e^{-i[\omega_d(\varepsilon k)/\varepsilon]t} \hat{\psi}_{+,0}^{w,\varepsilon}(k) - e^{-i[\omega_l(\varepsilon k)/\varepsilon]t} \hat{\psi}_{+,0}^{u,\varepsilon}(k) \right|^2 dk \\ &= \sup_{t \leq T/\varepsilon^2} \int_{\mathbb{R}^d} \left| e^{-i[\omega_d(\varepsilon k) - \omega_l(\varepsilon k)]t/\varepsilon} - 1 \right|^2 |\hat{\psi}_{+,0}^{w,\varepsilon}(k)|^2 dk \\ &\quad + \sup_{t \leq T/\varepsilon^2} \int_{\mathbb{R}^d} \left| e^{-i\omega_l(\varepsilon k)t/\varepsilon} \right|^2 |\hat{\psi}_{+,0}^{w,\varepsilon}(k) - \hat{\psi}_{+,0}^{u,\varepsilon}(k)|^2 dk \leq C\varepsilon^2. \end{aligned}$$

In the last step we used $\sup_{t \leq T/\varepsilon^2} [\omega_d(\varepsilon k) - \omega_l(\varepsilon k)]t/\varepsilon \leq C\varepsilon$ for every k in the compact set S_ψ , which is a consequence of (3.4). For the second term we used that initial values are nearby due to (3.10). The property for w and u translates into a corresponding property for ψ^w and ψ^u . The calculation for “-” is accordingly.

In order to obtain (3.12), we use the reconstruction formulas, see (2.7). We calculate for the first term

$$\begin{aligned} & \sup_{t \leq T/\varepsilon^2} \left\| \frac{\omega_0(\varepsilon \cdot)}{\varepsilon} [\hat{w}^\varepsilon(\cdot, t) - \hat{u}^\varepsilon(\cdot, t)] \right\|_{L^2(\mathbb{R}^d)}^2 \\ &= \sup_{t \leq T/\varepsilon^2} \frac{1}{2} \left\| \frac{\omega_0(\varepsilon \cdot)}{\varepsilon} \left[\frac{\varepsilon}{\omega_d(\varepsilon \cdot)} (\hat{\psi}_+^{w,\varepsilon} + \hat{\psi}_-^{w,\varepsilon}) - \frac{\varepsilon}{\omega_l(\varepsilon \cdot)} (\hat{\psi}_+^{u,\varepsilon} + \hat{\psi}_-^{u,\varepsilon}) \right] \right\|_{L^2(\mathbb{R}^d)}^2. \end{aligned}$$

The expression coming from the contributions $\hat{\psi}_+^{w,\varepsilon}$ and $\hat{\psi}_+^{u,\varepsilon}$ is, up to the factor $1/2$,

$$\begin{aligned} & \sup_{t \leq T/\varepsilon^2} \left\| \omega_0(\varepsilon \cdot) \left[\frac{\hat{\psi}_+^{w,\varepsilon}}{\omega_d(\varepsilon \cdot)} - \frac{\hat{\psi}_+^{u,\varepsilon}}{\omega_l(\varepsilon \cdot)} \right] \right\|_{L^2(\mathbb{R}^d)}^2 \\ &= \sup_{t \leq T/\varepsilon^2} \left\| \frac{\omega_0(\varepsilon \cdot)}{\omega_l(\varepsilon \cdot) \omega_d(\varepsilon \cdot)} (\omega_l(\varepsilon \cdot) - \omega_d(\varepsilon \cdot)) \hat{\psi}_+^{w,\varepsilon} + \frac{\omega_0(\varepsilon \cdot)}{\omega_l(\varepsilon \cdot)} (\hat{\psi}_+^{w,\varepsilon} - \hat{\psi}_+^{u,\varepsilon}) \right\|_{L^2(\mathbb{R}^d)}^2 \leq C\varepsilon^2. \end{aligned}$$

We exploited in the last step (3.4), which provides $|\omega_l(\varepsilon \cdot) - \omega_d(\varepsilon \cdot)| \leq C\varepsilon^4$; this implies that the first part is of order ε^2 . The second part is of order ε by (3.11). We make use of the fact that only k -values in the compact set S_ψ must be considered. The argument for the contributions of $\hat{\psi}_-^{w,\varepsilon}$ and $\hat{\psi}_-^{u,\varepsilon}$ is analogous.

For the estimate of time derivatives in (3.12), we use the reconstruction formulas $\partial_t \hat{u}^\varepsilon(k, t) = -i(\hat{\psi}_+^{u,\varepsilon} - \hat{\psi}_-^{u,\varepsilon})/\sqrt{2}$ and $\partial_t \hat{w}^\varepsilon(k, t) = -i(\hat{\psi}_+^{w,\varepsilon} - \hat{\psi}_-^{w,\varepsilon})/\sqrt{2}$. Relation (3.11) provides immediately the second part of (3.12). $\square$

4 Ring solutions and profile equations

The aim of this section is to derive a profile equation for the dispersive time scale. We will see that the evolution of a wave profile is given by the linearized KdV equation (1.4), which reads

$$\partial_\tau V^\varepsilon(z, \tau; q) = b(q) \partial_z^3 V^\varepsilon(z, \tau; q). \quad (4.1)$$

The profile is a function $V^\varepsilon = V^\varepsilon(z, \tau; q)$, the independent variables of the partial differential equation (4.1) are the time variable $\tau \in [0, T]$ and the spatial variable $z \in \mathbb{R}$. We consider a rescaled time variable, $t = \tau/\varepsilon^2$.

The vector $q \in S^{d-1}$ denotes the direction that is investigated. The parameter $\varepsilon > 0$ does not appear in the evolution equation, it enters only through the initial values. The coefficient $b(q) \in \mathbb{R}$ a constant that is obtained from an expansion of the dispersion relation. It is chosen in such a way that the lattice dispersion relation is approximated to third order: As $\tilde{k} \rightarrow 0$,

$$\omega_l(\tilde{k}) - [c|\tilde{k}| + b(\tilde{k})] = o(|\tilde{k}|^3). \quad (4.2)$$

Finally, the precise meaning of V^ε and z must be defined. We consider solutions u^ε to the lattice wave equation (1.1). If initial data are given by ε -independent functions in $L^2(\mathbb{R}^d)$, then most of the energy is initially contained in a ball $B_R(0) \subset \mathbb{R}^d$ for large $R > 0$. After some large time $t = \tau/\varepsilon^2$ for $\tau > 0$, the main contribution to the total energy comes from a ring $\{x \in \mathbb{R}^d : ||x| - ct| \leq R\}$ for large $R > 0$. Since we can expect the wave profile to be essentially independent of the angular variable, it is given by a function in z , where $z = |x| - ct$ is the relative radial position in the ring. We claim that the profile of u^ε is given by a solution V^ε of (4.1).

To formulate a mathematical statement, we now reverse the approach. We solve the one-dimensional linearized KdV equation (4.1) with appropriate initial data to obtain the function $V^\varepsilon(z, \tau, q)$. Using the shape function V^ε , we construct a function $v^\varepsilon(x, t)$ in such a way that the above interpretation of V^ε is respected. To be precise, the function v^ε is constructed from V^ε with the formula

$$v^\varepsilon(x, t) := \frac{1}{|x|^{(d-1)/2}} V^\varepsilon \left(|x| - ct, \varepsilon^2 t; \frac{x}{|x|} \right). \quad (4.3)$$

The prefactor $r^{-(d-1)/2}$ is introduced in order to keep the L^2 -norm of the function v^ε always of order 1. Our aim is to compare the reconstructed approximate solution v^ε with the original solution u^ε . Our result will be that the two functions u^ε and v^ε are comparable to leading order in ε .

Initial values and further notation. The initial values V_0^ε are extracted from the initial values u_0 of the lattice equation, but this construction is dimension dependent. We solve the evolution equation (4.1) with an initial condition of the form

$$V^\varepsilon(z, 0; q) = V_0^\varepsilon(z; q). \quad (4.4)$$

The vector $q \in S^{d-1}$ denotes a direction. In space dimension $d = 1$, we have to consider only two directions, $q \in S^0 = \{+1, -1\}$.

Reducing to $\hat{\psi}_{-,0}^{u,\varepsilon} = 0$, introducing Q_ε^u and Q_ε^v . In the next subsections, we will only consider the case of initial values with $\psi_{-,0}^{u,\varepsilon} = 0$. Such initial values occur if $i\partial_t \hat{u}^\varepsilon(k) = \omega_l(\varepsilon k)/\varepsilon \hat{u}^\varepsilon(k)$ in $t = 0$ holds for every k . The opposite case (i.e. $\psi_{+,0}^{u,\varepsilon} = 0$) can be treated in an analogous way. Since the equations are linear, one can combine the two results, hence the approximation results are also valid for general initial data.

Once we have decided to study $\psi_{-,0}^{u,\varepsilon} = 0$, we know that every solution in Fourier space has a fast time dependence given by the factor $e^{-ic|k|\tau/\varepsilon^2}$, where c is the wave-speed in $\omega_0(k) = c|k|$. In order to compensate this factor, we will compare in the following the two expressions

$$Q_\varepsilon^v(k, \tau) := e^{ic|k|\tau/\varepsilon^2} \hat{v}^\varepsilon(k, \tau/\varepsilon^2) \quad \text{and} \quad Q_\varepsilon^u(k, \tau) := e^{ic|k|\tau/\varepsilon^2} \hat{u}^\varepsilon(k, \tau/\varepsilon^2). \quad (4.5)$$

Remark: The factor $e^{ic|k|\tau/\varepsilon^2}$ can be interpreted as follows: In order to investigate the two solutions $\hat{v}^\varepsilon$ and $\hat{u}^\varepsilon$ at time t , we solve the linear wave equation backwards, and compare the corresponding values in $t = 0$.

From the explicit solution formula (2.6) and the reconstruction rule (2.7) we find

$$\begin{aligned} Q_\varepsilon^u(k, \tau) &= e^{ic|k|\tau/\varepsilon^2} \hat{u}^\varepsilon(k, \tau/\varepsilon^2) \\ &= e^{ic|k|\tau/\varepsilon^2} \frac{\varepsilon}{\omega_l(\varepsilon \cdot) \sqrt{2}} \left(\hat{\psi}_+^{u,\varepsilon}(k, \tau/\varepsilon^2) + \hat{\psi}_-^{u,\varepsilon}(k, \tau/\varepsilon^2) \right) \\ &= e^{ic|k|\tau/\varepsilon^2} \frac{\varepsilon}{\omega_l(\varepsilon \cdot) \sqrt{2}} e^{-i[\omega_l(\varepsilon k)/\varepsilon]\tau/\varepsilon^2} \hat{\psi}_{+,0}^u(k) \\ &= e^{-i[\omega_l(\varepsilon k)/\varepsilon - c|k|]\tau/\varepsilon^2} \hat{u}_0^\varepsilon(k). \end{aligned}$$

By the choice of the KdV-constant $b(q)$ in (4.2) we can compare $\omega_l(\varepsilon k)/\varepsilon - c|k|$ with $\varepsilon^2 b(k)$. We find

$$Q_\varepsilon^u(k, \tau) = e^{-ib(k)\tau} \hat{u}_0^\varepsilon(k) + F_\varepsilon(k, \tau) \quad (4.6)$$

with an error function that satisfies

$$|F_\varepsilon(k, \tau)| = \left| e^{-i[\omega_l(\varepsilon k) - c|\varepsilon k| - ib(\varepsilon k)]\tau/\varepsilon^3} - 1 \right| |\hat{u}_0^\varepsilon(k)|. \quad (4.7)$$

Due to (4.2) there holds $[\omega_l(\varepsilon k) - c|\varepsilon k| - ib(\varepsilon k)]\tau/\varepsilon^3 = o(1)$, uniformly in $\tau \in [0, T]$ and uniformly in k for k contained in a compact subset of $\mathbb{R}^d$.

4.1 The KdV profile in one dimension

In space dimension $d = 1$, the initial data V_0^ε can be defined easily through u_0 . More precisely, we define the Fourier transform of V_0^ε through the Fourier transform of u_0 . The Fourier transform of $V_0^\varepsilon(\cdot, q)$ is denoted as $\hat{V}_0^\varepsilon(\cdot; q)$, the primal variable is $z \in \mathbb{R}$, the dual variable is $\xi \in \mathbb{R}$. We set

$$\begin{aligned} \hat{V}_0^\varepsilon(\xi; +1) &:= \begin{cases} \hat{u}_0^\varepsilon(\xi) & \text{for } \xi > 0, \\ 0 & \text{else,} \end{cases} \\ \hat{V}_0^\varepsilon(\xi; -1) &:= \begin{cases} \hat{u}_0^\varepsilon(-\xi) & \text{for } \xi > 0, \\ 0 & \text{else.} \end{cases} \end{aligned} \quad (4.8)$$

In space dimension $d = 1$ the reconstruction formula (4.3) reads

$$v^\varepsilon(x, t) := \begin{cases} V^\varepsilon(|x| - ct, \varepsilon^2 t; +1) & \text{for } x > 0, \\ V^\varepsilon(|x| - ct, \varepsilon^2 t; -1) & \text{for } x < 0. \end{cases}$$

We can now calculate the complex number $Q_\varepsilon^v(k, \tau)$, using $t = \tau/\varepsilon^2$ to shorten the formulas. We use (4.5) in the first step, the Fourier transform in the second, and the reconstruction formula in the third step.

$$\begin{aligned} Q_\varepsilon^v(k, \tau) &= e^{ic|k|t} \hat{v}^\varepsilon(k, t) = e^{ic|k|t} \int_{-\infty}^{\infty} e^{-ik \cdot x} v^\varepsilon(x, t) dx \\ &= e^{ic|k|t} \sum_{q=\pm 1} \int_0^\infty e^{-ik \cdot qr} V^\varepsilon(r - ct, \varepsilon^2 t; q) dr \\ &= e^{ic|k|t} \sum_{q=\pm 1} \int_{\mathbb{R}} e^{-ik \cdot qct} e^{-ik \cdot qz} V^\varepsilon(z, \varepsilon^2 t; q) dz + G_\varepsilon(k, \tau) \\ &= e^{ic|k|t} \sum_{q=\pm 1} e^{-ik \cdot qct} \hat{V}^\varepsilon(k \cdot q, \varepsilon^2 t; q) + G_\varepsilon(k, \tau), \end{aligned}$$

where we introduced the error function

$$G_\varepsilon(k, \tau) := -e^{ic|k|\tau/\varepsilon^2} \sum_{q=\pm 1} \int_{-\infty}^0 e^{-ik \cdot qr} V^\varepsilon(r - c\tau/\varepsilon^2, \tau; q) dr. \quad (4.9)$$

We now exploit that $\hat{V}_0^\varepsilon$ is supported on the positive half-line $[0, \infty)$. This implies that the evolution by the linearized KdV equation (4.1) is given in Fourier space by the exponential factor $e^{-ib(|\xi|q)\tau}$. We obtain

$$Q_\varepsilon^v(k, \tau) = e^{ic|k|t} \sum_{q=\pm 1} e^{-ik \cdot qct} e^{-ib(|k|q)\varepsilon^2 t} \hat{V}_0^\varepsilon(k \cdot q; q) + G_\varepsilon(k, \tau). \quad (4.10)$$

We next expand the sum, using $q = +k/|k|$ and $q = -k/|k|$. For $q = k/|k|$, two exponential factors cancel. The second term vanishes since $\hat{V}_0^\varepsilon$ vanishes for negative first arguments.

$$Q_\varepsilon^v(k, \tau) = e^{-ib(k)\varepsilon^2 t} \hat{V}_0^\varepsilon(|k|; k/|k|) \quad (4.11)$$

$$\begin{aligned} &+ e^{2ic|k|t} e^{-ib(-k)\varepsilon^2 t} \hat{V}_0^\varepsilon(-|k|; -k/|k|) + G_\varepsilon(k, \tau) \\ &= e^{-ib(k)\tau} \hat{V}_0^\varepsilon(|k|; k/|k|) + G_\varepsilon(k, \tau) \\ &= e^{-ib(k)\tau} \hat{u}_0^\varepsilon(k) + G_\varepsilon(k, \tau), \end{aligned} \quad (4.12)$$

where in the last equality we used the choice of initial data $\hat{V}_0^\varepsilon$ in (4.8).

We calculated $Q_\varepsilon^u(k, \tau)$ in (4.6) and $Q_\varepsilon^v(k, \tau)$ in (4.12). We found the same leading order term in both calculations, namely $e^{-ib(k)\tau} \hat{u}_0^\varepsilon(k)$. The difference is

$$|Q_\varepsilon^u(k, \tau) - Q_\varepsilon^v(k, \tau)| \leq |F_\varepsilon(k, \tau)| + |G_\varepsilon(k, \tau)|. \quad (4.13)$$

Theorem 4.1 (KdV profile equation in one space dimension). *We consider $d = 1$ and a sequence of initial values $(u_0^\varepsilon, u_1^\varepsilon)$ with $\hat{u}_1^\varepsilon(\cdot) = -i\varepsilon^{-1}\omega_l(\varepsilon)\hat{u}_0^\varepsilon(\cdot)$ on $\mathbb{R}$. We assume $\hat{u}_0^\varepsilon \rightarrow \hat{u}_0$ strongly in $L^2(\mathbb{R})$ as $\varepsilon \rightarrow 0$ for some limit $\hat{u}_0$. We furthermore assume that all functions $\hat{u}_0^\varepsilon(\cdot)$ have support in some bounded domain $S_\psi \subset \mathbb{R}$.*

Let $u^\varepsilon(., t)$ be the solution to the lattice equation (1.1), let V^ε be the solution to the KdV-equation (4.1) with initial values (4.8) and coefficients c and b from (4.2). Let $v^\varepsilon(., t)$ be given by the reconstruction formula (4.3). Then, for every $\tau \in (0, T]$,

$$\|\hat{u}^\varepsilon(., \tau/\varepsilon^2) - \hat{v}^\varepsilon(., \tau/\varepsilon^2)\|_{L^2(\mathbb{R})} \rightarrow 0 \quad (4.14)$$

as $\varepsilon \rightarrow 0$.

Proof. The error term G_ε of (4.9) can be written with a change of variables and with the characteristic function $\mathbf{1}_{\{s|s \leq -ct\}}$ in the form (we use $t = \tau/\varepsilon^2$)

$$\begin{aligned} G_\varepsilon(k, \tau) &= - \sum_{q=\pm 1} e^{i|k|ct - ik \cdot qct} \int_{-\infty}^{\infty} e^{-ik \cdot qs} \mathbf{1}_{\{s \leq -ct\}}(s) V^\varepsilon(s, \tau; q) ds \\ &= - \sum_{q=\pm 1} e^{i|k|ct - ik \cdot qct} \mathcal{F}(\mathbf{1}_{\{s \leq -ct\}}(\cdot) V^\varepsilon(\cdot, \tau; q))(k \cdot q). \end{aligned}$$

We obtain

$$\|G_\varepsilon(\cdot, \tau)\|_{L^2(dk)}^2 \leq 2 \sum_{q=\pm 1} \|\mathbf{1}_{\{s \leq -ct\}}(\cdot) V^\varepsilon(s, \tau; q)\|_{L^2(ds)}^2. \quad (4.15)$$

The solution V^ε of the KdV equation can be expressed in Fourier space. The strong convergence of the (Fourier transform of the) initial values in (4.8) allows to compare the solution V^ε with its formal limit V^0 ,

$$\begin{aligned} \|V^\varepsilon(s, \tau; q) - V^0(s, \tau; q)\|_{L^2(ds)}^2 &= \|\hat{V}^\varepsilon(\xi, \tau; q) - \hat{V}^0(\xi, \tau; q)\|_{L^2(d\xi)}^2 \\ &= \left\| e^{-ib(|\xi|q)\tau} \left(\hat{V}_0^\varepsilon(\xi; q) - \hat{V}_0^0(\xi; q) \right) \right\|_{L^2(d\xi)}^2 \rightarrow 0, \end{aligned}$$

uniformly in $\tau \in [0, T]$. The triangle inequality allows to continue from (4.15):

$$\begin{aligned} \|G_\varepsilon(\cdot, \tau)\|_{L^2(dk)}^2 &\leq 2 \sum_{q=\pm 1} \|\mathbf{1}_{\{s \leq -c\tau/\varepsilon^2\}}(s) V^\varepsilon(s, \tau; q)\|_{L^2(ds)}^2 \\ &\leq 4 \sum_{q=\pm 1} \|\mathbf{1}_{\{s \leq -c\tau/\varepsilon^2\}}(s) (V^\varepsilon(s, \tau; q) - V^0(s, \tau; q))\|_{L^2(ds)}^2 \\ &\quad + 4 \sum_{q=\pm 1} \|\mathbf{1}_{\{s \leq -c\tau/\varepsilon^2\}}(s) V^0(s, \tau; q)\|_{L^2(ds)}^2 \rightarrow 0 \end{aligned}$$

for every $\tau \in (0, T]$, since the integral of the (ε -independent) L^1 -function $|V^0(\cdot, \tau; q)|^2$ over $(-\infty, -c\tau/\varepsilon^2)$ vanishes in the limit $\varepsilon \rightarrow 0$.

We have obtained that both error terms F_ε and G_ε vanish in the limit $\varepsilon \rightarrow 0$ and conclude from (4.13), for every $\tau \in (0, T]$,

$$\begin{aligned} \|\hat{u}^\varepsilon(., \tau/\varepsilon^2) - \hat{v}^\varepsilon(., \tau/\varepsilon^2)\|_{L^2(dk)}^2 &= \|e^{ic|k|t} \hat{u}^\varepsilon(., \tau/\varepsilon^2) - e^{ic|k|t} \hat{v}^\varepsilon(., \tau/\varepsilon^2)\|_{L^2(dk)}^2 \\ &= \|Q_\varepsilon^u(\cdot, \tau) - Q_\varepsilon^v(\cdot, \tau)\|_{L^2(dk)}^2 \\ &\leq 2\|F_\varepsilon(\cdot, \tau)\|_{L^2(dk)}^2 + 2\|G_\varepsilon(\cdot, \tau)\|_{L^2(dk)}^2 \rightarrow 0 \end{aligned}$$

as $\varepsilon \rightarrow 0$. This was the claim in (4.14). $\square$

Remark: The two error terms satisfy additionally bounds that are uniform in τ : $\sup_{\tau \in [0, T]} \|F_\varepsilon(\cdot, \tau)\|_{L^2(dk)} \leq C$ and $\sup_{\tau \in [0, T]} \|G_\varepsilon(\cdot, \tau)\|_{L^2(dk)} \leq C$. We therefore also obtain the convergence in (4.14) in an integral sense, e.g.:

$$\int_0^T \|\hat{u}^\varepsilon(., \tau/\varepsilon^2) - \hat{v}^\varepsilon(., \tau/\varepsilon^2)\|_{L^2(\mathbb{R})} d\tau = o(1). \quad (4.16)$$

4.2 Ring solutions in arbitrary dimension

The Fourier transform of the reconstructed solution v^ε can be calculated in arbitrary dimension $d \geq 1$. We assume in this calculation that $\hat{V}_0^\varepsilon(\cdot; q)$ is supported on $\xi \in [0, \infty)$. In this case $\xi^3 = |\xi|^3$ for all ξ in the support and we can write (as in the one-dimensional case) the solution of the linearized KdV equation in Fourier space as

$$\hat{V}^\varepsilon(\xi, \tau; q) = e^{-ib(|\xi|q)\tau} \hat{V}_0^\varepsilon(\xi; q). \quad (4.17)$$

We can calculate the Fourier transform of the reconstruction $\hat{v}^\varepsilon$ in arbitrary dimension. In the following calculation we use: 1. The definition of the Fourier transform. 2. Polar coordinates $x = rq$, $r > 0$, $q \in S^{d-1}$. 3. The reconstruction formula (4.3). 4. The integral has its main contributions for the radial component $r \approx ct$; the replacement introduces an error G_ε .

$$\begin{aligned} \hat{v}^\varepsilon(k, t) &= \int_{\mathbb{R}^d} e^{-ik \cdot x} v^\varepsilon(x, t) dx \\ &= \int_{S^{d-1}} \int_0^\infty e^{-ik \cdot qr} v^\varepsilon(rq, t) r^{d-1} dr dS(q) \\ &= \int_{S^{d-1}} \int_0^\infty e^{-ik \cdot qr} V^\varepsilon(r - ct, \tau; q) r^{(d-1)/2} dr dS(q) \\ &= \int_{S^{d-1}} \int_{-\infty}^\infty e^{-ik \cdot qr} V^\varepsilon(r - ct, \tau; q) |ct|^{(d-1)/2} dr dS(q) + G_\varepsilon(k, \tau) \end{aligned} \quad (4.18)$$

with the error function

$$G_\varepsilon(k, \tau) := \int_{S^{d-1}} \int_{-\infty}^\infty e^{-ik \cdot qr} V^\varepsilon(r - ct, \tau; q) [r^{(d-1)/2} \mathbf{1}_{r \geq 0} - |ct|^{(d-1)/2}] dr dS(q). \quad (4.19)$$

5 Approximation result in two dimensions

We now consider the case of space dimension $d = 2$ in more detail. To simplify formulas, we use ε -independent initial data $\hat{u}_0^\varepsilon = \hat{u}_0$ with bounded support $S_\psi \subset \mathbb{R}^2$. We note that the (discrete) Fourier expansion of the function $\hat{u}_0$ (understood as a periodic function on the Brillouin zone) provides the initial data u_0^ε . Since we additionally want to work with $\psi_{-,0}^{u,\varepsilon} = 0$, we consider $\hat{u}_1^\varepsilon(\cdot) = -i\omega_l(\varepsilon)/\varepsilon \hat{u}_0^\varepsilon(\cdot)$ on $\mathbb{R}^2$.

We next choose initial values $V_0^\varepsilon(z; q) = V_0(z; q)$ for the linearized KdV equation (4.1). Loosely speaking, we want to choose initial values

$$\hat{V}_0(\xi; q) \approx z_0^{-1} \sqrt{\xi} \hat{u}_0(\xi q) \forall \xi > 0 \quad \text{with} \quad z_0 := \sqrt{\pi}(1 + i). \quad (5.1)$$

The essential part is the factor $\sqrt{\xi}$, which was not present in the one-dimensional case, compare (4.8).

In order to obtain convergence results, we have to replace the square-root function by a differentiable function W_ρ , where $\rho > 0$ is a small parameter. We demand:

$$W_\rho \in C^1(\mathbb{R}; \mathbb{R}), \quad W_\rho(\xi) = 0 \quad \forall \xi \leq 0, \quad W_\rho(\xi) = \sqrt{\xi} \quad \forall \xi \geq \rho/2. \quad (5.2)$$

Using the function W_ρ , we define the initial data for the linearized KdV equation with the help of the initial data u_0^ε by setting

$$\hat{V}_0^\varepsilon(\xi; q) := \begin{cases} z_0^{-1} W_\rho(\xi) \hat{u}_0^\varepsilon(\xi q) & \text{for } \xi > 0, \\ 0 & \text{else.} \end{cases} \quad (5.3)$$

5.1 Pointwise convergence

We want to exploit the calculation (4.18). As a first step, we have to show smallness of the error term G_ε from (4.19), which, for $d = 2$, reads

$$G_\varepsilon(k, \tau) = \int_{S^1} \int_{-\infty}^{\infty} e^{-ik \cdot qr} V^\varepsilon(r - ct, \tau; q) [r^{1/2} \mathbf{1}_{r \geq 0} - (ct)^{1/2}] dr dS(q). \quad (5.4)$$

Lemma 5.1 (Smallness of the error G_ε). *We consider $d = 2$ and assume that the solution $V^\varepsilon(s, \tau; q)$ to the linearized KdV equation (4.1) with initial values (5.3) has the following decay property for constants $C, \alpha, \varepsilon_0 > 0$:*

$$|V^\varepsilon(s, \tau; q)| \leq C(1 + |s|)^{-2-\alpha}, \quad (5.5)$$

for all $\tau \in [0, T]$, $q \in S^1$, $\varepsilon \in (0, \varepsilon_0)$. Then the error function G_ε of (5.4) satisfies, for every $\tau \in (0, T]$,

$$G_\varepsilon(k, \tau) \rightarrow 0 \quad (5.6)$$

as $\varepsilon \rightarrow 0$, uniformly in $k \in \mathbb{R}^2$.

Proof of Lemma 5.1. We choose $\tau \in (0, T]$. We observe that S^1 has the finite measure 2π and that $e^{-ik \cdot qr}$ has norm 1. Therefore, in order to verify (5.6), it suffices to show (we substitute $r = c\tau/\varepsilon^2 + s$)

$$\int_{-\infty}^{\infty} |V^\varepsilon(s, \tau; q)| |(c\tau/\varepsilon^2 + s)^{1/2} \mathbf{1}_{s \geq -c\tau/\varepsilon^2} - |c\tau/\varepsilon^2|^{1/2}| ds \rightarrow 0, \quad (5.7)$$

uniformly in $q \in S^1$. We decompose the integral into two parts, distinguishing $|s| \leq \delta/\varepsilon$ and $|s| > \delta/\varepsilon$, $\delta > 0$ is chosen below. We only consider ε -values with $c\tau/\varepsilon > \delta$.

The integral over $\{|s| \leq \delta/\varepsilon\}$ is estimated using $|(c\tau/\varepsilon^2 + s)^{1/2} - |c\tau/\varepsilon^2|^{1/2}| = \varepsilon^{-1} |(c\tau + s\varepsilon^2)^{1/2} - |c\tau|^{1/2}| \leq C s \varepsilon \leq C\delta$:

$$\begin{aligned} & \int_{\{|s| \leq \delta/\varepsilon\}} |V^\varepsilon(s, \tau; q)| |(c\tau/\varepsilon^2 + s)^{1/2} \mathbf{1}_{s \geq -c\tau/\varepsilon^2} - |c\tau/\varepsilon^2|^{1/2}| ds \\ & \leq C\delta \int_{-\infty}^{\infty} |V^\varepsilon(s, \tau; q)| ds \leq C\delta. \end{aligned}$$

The other integral concerns large values of $|s|$, we consider $\{|s| > \delta/\varepsilon\}$. The integral is treated with the crude estimate $|(c\tau/\varepsilon^2 + s)^{1/2} \mathbf{1}_{s \geq -c\tau/\varepsilon^2} - |c\tau/\varepsilon^2|^{1/2}| \leq C(\varepsilon^{-1} + |s|^{1/2})$. Assumption (5.5) allows to calculate

$$\begin{aligned} & \int_{\{|s| > \delta/\varepsilon\}} |V^\varepsilon(s, \tau; q)| |(c\tau/\varepsilon^2 + s)^{1/2} \mathbf{1}_{s \geq -c\tau/\varepsilon^2} - |c\tau/\varepsilon^2|^{1/2}| ds \\ & \leq C \int_{\{|s| > \delta/\varepsilon\}} (\varepsilon^{-1} + |s|^{1/2}) |V^\varepsilon(s, \tau; q)| ds \\ & \leq C \int_{\{|s| > \delta/\varepsilon\}} (\varepsilon^{-1} + |s|^{1/2}) |s|^{-2-\alpha} ds \leq C\varepsilon^{-1}(\varepsilon/\delta)^{1+\alpha} + C(\varepsilon/\delta)^{1/2+\alpha}. \end{aligned}$$

Choosing first $\delta > 0$ to have smallness of the first integral, and choosing then ε_0 , we obtain smallness of both integrals. This yields (5.6). $\square$

We now continue the calculation (4.18) of $\hat{v}^\varepsilon$ for $d = 2$. Substituting once more $r = ct + s$, we recognize the one-dimensional Fourier transform of V^ε . This allows to insert the solution formula (4.17) in the last equality:

$$\begin{aligned}\hat{v}^\varepsilon(k, t) &= \int_{S^1} \int_{-\infty}^{\infty} e^{-ik \cdot q(ct+s)} V^\varepsilon(s, \tau; q) |ct|^{1/2} ds dS(q) + G_\varepsilon(k, \tau) \\ &= \int_{S^1} e^{-ik \cdot qct} \hat{V}^\varepsilon(k \cdot q, \tau; q) |ct|^{1/2} dS(q) + G_\varepsilon(k, \tau) \\ &= \int_{S^1} e^{-ik \cdot qct} e^{-ib((k \cdot q)q)\tau} \hat{V}_0^\varepsilon(k \cdot q; q) |ct|^{1/2} dS(q) + G_\varepsilon(k, \tau).\end{aligned}$$

With this expression for $\hat{v}^\varepsilon(k, t)$ we can, as in the one-dimensional case, evaluate the function $Q_\varepsilon^v(k, \tau)$ of (4.5). We insert the initial conditions $\hat{V}_0^\varepsilon$ from (5.3) and find

$$\begin{aligned}Q_\varepsilon^v(k, \tau) &= e^{ic|k|\tau/\varepsilon^2} \hat{v}^\varepsilon(k, \tau/\varepsilon^2) \\ &= \int_{S^1 \cap \{k \cdot q > 0\}} e^{i(|k| - k \cdot q)c\tau/\varepsilon^2} e^{-ib((k \cdot q)q)\tau} z_0^{-1} W_\rho(k \cdot q) \hat{u}_0^\varepsilon((k \cdot q)q) \frac{\sqrt{c\tau}}{\varepsilon} dS(q) \\ &\quad + e^{ic|k|\tau/\varepsilon^2} G_\varepsilon(k, \tau).\end{aligned}$$

It remains to compare this expression with $Q_\varepsilon^u(k, \tau) \approx e^{-ib(k)\tau} \hat{u}_0^\varepsilon(k)$ from (4.6). With this aim we write the above integral in the form

$$Q_\varepsilon^v(k, \tau) = z_0^{-1} \int_{S^1} \Phi_\varepsilon(q; k, \tau) \varphi^\varepsilon(q; k, \tau) dS(q) + e^{ic|k|\tau/\varepsilon^2} G_\varepsilon(k, \tau) \quad (5.8)$$

with the two functions

$$\Phi_\varepsilon(q; k, \tau) := e^{i(|k| - k \cdot q)c\tau/\varepsilon^2} W_\rho(k \cdot q) \frac{\sqrt{c\tau}}{\varepsilon} \mathbf{1}_{\{k \cdot q > 0\}}, \quad (5.9)$$

$$\varphi^\varepsilon(q; k, \tau) := e^{-ib((q \cdot k)k/|k|)\tau} \hat{u}_0^\varepsilon((k \cdot q)q). \quad (5.10)$$

Loosely speaking, we will verify the following: If we consider Φ_ε as a function in $q \in S^1$, this sequence is a Dirac-sequence for the point $q_0 = k/|k|$. As a result, $Q_\varepsilon^v(k, \tau)$ from (5.8) can be compared with $\varphi^\varepsilon(q_0; k, \tau)$, which is essentially $Q_\varepsilon^u(k, \tau)$ from (4.6).

Since we have to evaluate the above expression for fixed k , it is no restriction to consider only a wave vector in the first coordinate direction, $k = k_1 e$. We abbreviate the subsequent calculation by setting $\eta = \varepsilon^2/(k_1 c\tau)$.

Lemma 5.2 (Dirac-sequence for $d = 2$). *We consider $k_1 \geq \rho$ such that $W_\rho(k_1) = \sqrt{k_1}$. On the 1-sphere $S^1 \subset \mathbb{R}^2 \equiv \mathbb{C}$, points are denoted as $q = (q_1, q_2) \in S^1$, the measure of integration is $dS(q) = d\mathcal{H}^1(q)$. We consider the following sequence of functions $\Phi^\eta(\cdot) : S^1 \rightarrow \mathbb{C}$:*

$$\Phi^\eta(q) = e^{i(1-q_1)/\eta} \frac{1}{\sqrt{\eta}} \frac{W_\rho(k_1 q_1)}{\sqrt{k_1}} \mathbf{1}_{\{q_1 > 0\}}. \quad (5.11)$$

Then there holds

$$\Phi^\eta(\cdot) \rightarrow z_0 \delta_{e_1}(\cdot) \quad \text{with} \quad z_0 = \sqrt{\pi}(1+i) \quad (5.12)$$

in the sense of distributions on S^1 as $\eta \rightarrow 0$.

Proof. Step 1: Simplification. Let $\varphi : S^1 \rightarrow \mathbb{R}$ be a smooth test-function. We parametrize the sphere with the map $(-\pi, \pi) \ni \theta \mapsto q = e^{i\theta} \in S^1$, making use of the natural identification $\mathbb{C} \equiv \mathbb{R}^2$. In particular, there holds $q_1 = \cos(\theta)$.

To abbreviate calculations, we modify the test-function and set

$$\tilde{\varphi}(\theta) := \frac{W_\rho(k_1 q_1)}{\sqrt{k_1}} \varphi(q) = \frac{W_\rho(k_1 \cos(\theta))}{\sqrt{k_1}} \varphi(e^{i\theta}) \quad \text{and} \quad \bar{\varphi}(\theta) := \tilde{\varphi}(\theta) + \tilde{\varphi}(-\theta).$$

The symmetrized variant $\bar{\varphi}$ allows to consider only integrals over $\theta \in (0, \pi/2)$. The expression of interest now reads

$$\begin{aligned} \langle \Phi^\eta, \varphi \rangle &= \int_{S^1} \Phi^\eta(q) \varphi(q) dS(q) = \int_{-\pi/2}^{\pi/2} e^{i(1-\cos(\theta))/\eta} \frac{1}{\sqrt{\eta}} \tilde{\varphi}(\theta) d\theta \\ &= \int_0^{\pi/2} e^{i(1-\cos(\theta))/\eta} \frac{1}{\sqrt{\eta}} \bar{\varphi}(\theta) d\theta. \end{aligned}$$

Step 2: Decomposition. We first calculate the contribution from the integration away from $\theta = 0$ (away from the point $q = e_1 \in S^1$). For arbitrary $\delta > 0$, we use a trivial extension of the integrand in order to recognize one term as a derivative:

$$\begin{aligned} \int_\delta^{\pi/2} e^{i(1-\cos(\theta))/\eta} \frac{1}{\sqrt{\eta}} \bar{\varphi}(\theta) d\theta &= \int_\delta^{\pi/2} e^{i(1-\cos(\theta))/\eta} \frac{i \sin(\theta)}{\eta} \sqrt{\eta} \frac{\bar{\varphi}(\theta)}{i \sin(\theta)} d\theta \\ &= \int_\delta^{\pi/2} \partial_\theta [e^{i(1-\cos(\theta))/\eta}] \sqrt{\eta} \frac{\bar{\varphi}(\theta)}{i \sin(\theta)} d\theta = \sqrt{\eta} O(1/\delta^2). \end{aligned} \quad (5.13)$$

In the last step we performed an integration by parts and the estimate $|\partial_\theta(1/\sin(\theta))| \leq 1/\sin^2(\theta) \leq C/\delta^2$. Our choice of $\delta > 0$ will ensure smallness of the error term in (5.13).

It remains to investigate the integral over small values of θ . We calculate with a Taylor expansion of $\bar{\varphi}$ in $\theta = 0$

$$\int_0^\delta e^{i(1-\cos(\theta))/\eta} \frac{1}{\sqrt{\eta}} \bar{\varphi}(\theta) d\theta = \bar{\varphi}(0) \int_0^\delta e^{i(1-\cos(\theta))/\eta} \frac{1}{\sqrt{\eta}} d\theta + \frac{O(\delta^3)}{\sqrt{\eta}}. \quad (5.14)$$

In the estimate for the error we used that $\bar{\varphi}$ is symmetric which provides $\bar{\varphi}'(0) = 0$, and that the interval of integration has length δ .

In view of (5.13) and (5.14) we choose $\delta := \eta^{1/5}$. With this choice, there holds $\sqrt{\eta}/\delta^2 = \eta^{1/2}\eta^{-2/5} = \eta^{1/10}$ and $\delta^3/\sqrt{\eta} = \eta^{3/5}\eta^{-1/2} = \eta^{1/10}$. This shows that the error terms in (5.13) and (5.14) are both of order $\eta^{1/10}$.

Step 3: Limiting integral. We note that the factor in front of the integral in (5.14) is $\bar{\varphi}(0) = 2\tilde{\varphi}(0) = 2W_\rho(k_1)/\sqrt{k_1}\varphi(e_1) = 2\varphi(e_1)$. It therefore only remains to evaluate the limit of the integral

$$I_\eta := \int_0^\delta e^{i(1-\cos(\theta))/\eta} \frac{1}{\sqrt{\eta}} d\theta. \quad (5.15)$$

The integral can be written with the substitution $z = (1 - \cos(\theta))/\eta$, leading to $d\theta = \eta/\sin(\theta) dz$ with the inverse function $\theta = \theta(z) := (1 - \cos)^{-1}(\eta z) = \sqrt{2z\eta} + O((z\eta)^{3/2})$. We find

$$I_\eta = \int_0^{(1-\cos(\delta))/\eta} e^{iz} \frac{\sqrt{\eta}}{\sin(\theta)} dz. \quad (5.16)$$

Regarding the domain of integration we find $R(\eta) := (1 - \cos(\delta))/\eta \approx \frac{1}{2}\delta^2/\eta = \frac{1}{2}\eta^{-3/5} \rightarrow \infty$ as $\eta \rightarrow 0$. On the other hand, all values of z in the domain of integration satisfy $z\eta \leq C\eta^{2/5}$ and hence $\theta = \sqrt{2z\eta} + O((z\eta)^{3/2}) \leq C\eta^{1/5}$. We may therefore develop $\sin(\theta)$,

$$\sin(\theta) = \theta + O(\theta^3) = \sqrt{2z\eta} + O((z\eta)^{3/2}).$$

We find, as $\eta \rightarrow 0$,

$$\begin{aligned} I_\eta &= \int_0^{R(\eta)} e^{iz} \frac{1}{\sin(\theta)/\sqrt{\eta}} dz = \frac{1}{\sqrt{2}} \int_0^{R(\eta)} e^{iz} \frac{1}{\sqrt{z} + O(z^{3/2}\eta)} dz \\ &\rightarrow \frac{1}{\sqrt{2}} \int_0^\infty e^{iz} \frac{1}{\sqrt{z}} dz = \sqrt{2} \int_0^\infty e^{ip^2} dp = \frac{1}{2}\sqrt{\pi}(1+i) = \frac{1}{2}z_0. \end{aligned} \quad (5.17)$$

In the last line we used the substitution $z = p^2$ and Fresnel integrals: For real and imaginary part there holds $\int_0^\infty \sin(x^2) dx = \int_0^\infty \cos(x^2) dx = \sqrt{\pi}/(2\sqrt{2})$. Regarding the convergence of the integrals: For fixed $z \leq R(\eta) \leq C\eta^{-3/5}$, there holds $\frac{1}{\sqrt{z} + O(z^{3/2}\eta)} \rightarrow \frac{1}{\sqrt{z}}$ as $\eta \rightarrow 0$, we therefore have pointwise convergence of the integrands. For large values of z we expand the fraction and find $1/(\sqrt{z} + O(z^{3/2}\eta)) = 1/\sqrt{z} + O(z^{1/2}\eta)$. Because of $\int_0^{R(\eta)} z^{1/2}\eta = O(R(\eta)^{3/2}\eta) = O(\eta^{-9/10}\eta) = O(\eta^{1/10})$, the integral of the error term is small. This justifies the limit in (5.17) and provides the claim of (5.12). $\square$

We collect our results in the following theorem on pointwise convergence.

Theorem 5.3 (Pointwise convergence of the Fourier transforms). *Let initial values be given by $(u_0^\varepsilon, u_1^\varepsilon)$ with $\hat{u}_0^\varepsilon = \hat{u}_0$ continuous and ε -independent and with $\hat{u}_1^\varepsilon(\cdot) = -i\omega_1(\varepsilon)/\varepsilon \hat{u}_0^\varepsilon(\cdot)$ on $\mathbb{R}^2$. We assume that $\hat{u}_0(\cdot)$ has support in the bounded domain $S_\psi \subset \mathbb{R}^2$. Let $u^\varepsilon(\cdot, t)$ be the solution to the lattice equation (1.1).*

Let V^ε be the solution to the KdV-equation (4.1) with initial values (5.3) for some parameter $\rho > 0$. We assume that V^ε satisfies the decay estimate (5.5). Let $v^\varepsilon(\cdot, t)$ be given by the reconstruction formula (4.3).

Then there holds

$$|\hat{u}^\varepsilon(k, \tau/\varepsilon^2) - \hat{v}^\varepsilon(k, \tau/\varepsilon^2)| \rightarrow 0 \quad (5.18)$$

as $\varepsilon \rightarrow 0$ for every $\tau \in (0, T]$ and for every $k \in \mathbb{R}^2$ with $|k| \geq \rho$.

Remark: The assumptions on the initial data and on the decay of V^ε are related: Smoothness of $\hat{u}_0$ and of $\hat{V}^0$ imply spatial decay properties of V^ε .

Proof. We first compare the two expressions $Q_\varepsilon^u(k, \tau)$ and $Q_\varepsilon^v(k, \tau)$ of (4.5). We have obtained a simplified expression for $Q_\varepsilon^u(k, \tau)$ in (4.6), which implies for $\varepsilon \rightarrow 0$

$$Q_\varepsilon^u(k, \tau) \rightarrow e^{-ib(k)\tau} \hat{u}_0(k). \quad (5.19)$$

For $Q_\varepsilon^v(k, \tau)$, we calculated a simplification in (5.8). The error term G_ε was estimated in Lemma 5.1 using (5.5). Lemma 5.2 can be used to calculate the limit of the integral in (5.8), since $\varphi^\varepsilon = \varphi$ is a continuous and ε -independent function. We emphasize that continuity of $\hat{u}_0^\varepsilon = \hat{u}_0$ is exploited here. We find, with $q_0 = k/|k|$,

$$Q_\varepsilon^v(k, \tau) \rightarrow \varphi(q_0; k, \tau) = e^{-ib((k \cdot q_0) q_0)\tau} \hat{u}_0((k \cdot q_0) q_0) = e^{-ib(k)\tau} \hat{u}_0(k). \quad (5.20)$$

The limits in (5.19) and (5.20) are identical.

This implies also the pointwise convergence of the Fourier transforms of the two solutions:

$$\begin{aligned} |\hat{u}^\varepsilon(k, \tau/\varepsilon^2) - \hat{v}^\varepsilon(k, \tau/\varepsilon^2)| &= |e^{ic|k|\tau/\varepsilon^2} \hat{u}^\varepsilon(k, \tau/\varepsilon^2) - e^{ic|k|\tau/\varepsilon^2} \hat{v}^\varepsilon(k, \tau/\varepsilon^2)| \\ &= |Q_\varepsilon^u(k, \tau) - Q_\varepsilon^v(k, \tau)| \rightarrow 0. \end{aligned}$$

This was the claim in (5.18). $\square$

Remark 5.4. Assume that $\hat{u}_0^\varepsilon = \hat{u}_0$ is a smooth function with bounded support. In this case all the functions $\hat{u}^\varepsilon$, $\hat{V}_0^\varepsilon$ from (5.3), and $\hat{V}^\varepsilon$ have compact support in k . Furthermore, the differentiability of $\hat{u}_0$ is inherited by the other functions and in particular by $\hat{V}^\varepsilon$. This, in turn, implies that the decay property (5.5) of V^ε is satisfied and hence (5.18) holds.

5.2 On strong convergence

The ultimate goal in the analysis of ring solutions in two dimensions is to show the strong convergence $u^\varepsilon - v^\varepsilon \rightarrow 0$ in $L^2(\mathbb{R}^2)$. This is a very challenging task.

Of course, since the Fourier transform is an isometry in L^2 , the strong convergence in physical space can follow from strong convergence in Fourier space, $\hat{u}^\varepsilon - \hat{v}^\varepsilon \rightarrow 0$ in $L^2(\mathbb{R}^2)$. We recall that we have the pointwise convergence of (5.18) at our disposal. The main difficulty in proving the strong convergence is the behavior of the Fourier transform $\hat{v}^\varepsilon$ in a neighborhood of $k = 0$. We recall that the pointwise convergence of (5.18) is only valid for $|k| > \rho$. Indeed, since we have chosen a regularized form for V_0^ε , modifying the function for $|k| \leq \rho$, we cannot expect any better pointwise convergence. The best result that we could obtain is therefore

$$\limsup_{\varepsilon \rightarrow 0} \|\hat{u}^\varepsilon(\cdot, \tau/\varepsilon^2) - \hat{v}^\varepsilon(\cdot, \tau/\varepsilon^2)\|_{L^2(\mathbb{R}^2)} \leq h(\rho), \quad (5.21)$$

where $h : [0, 1) \rightarrow [0, \infty)$ is some function with $h(\rho) \rightarrow 0$ for $\rho \rightarrow 0$.

Comments on the convergence (5.21). We assume once more that $\hat{u}_0^\varepsilon = \hat{u}_0$ is a smooth function with bounded support. Regarding large values of $|k|$ we note that the smoothness of V_0 implies differentiability properties of V^ε and hence of v^ε . These, in turn, imply decay properties of $\hat{v}^\varepsilon$. Since, on the other hand, $\hat{u}^\varepsilon$ has bounded support, we conclude that $|\hat{u}^\varepsilon(k, \tau/\varepsilon^2) - \hat{v}^\varepsilon(k, \tau/\varepsilon^2)|$ has also decay properties for $|k| \rightarrow \infty$.

An inspection of the proofs reveals that the convergence is also uniform on compact subsets of $k \in \mathbb{R}^2 \setminus B_\rho(0)$. All these considerations suggest that (5.21) holds at least in $L^2(\mathbb{R}^2 \setminus B_\rho(0))$.

The behavior of $\hat{v}^\varepsilon$ for small values of $|k|$ is much more intricate; a first hint of this fact is that for ring solutions we necessarily have $\hat{v}^\varepsilon(0) = \int_{\mathbb{R}^2} v^\varepsilon(x) dx \rightarrow \infty$ as $\varepsilon \rightarrow 0$. Nevertheless, a positive result can be expected, as the subsequent lemma suggests: The function w^ε is constructed as a ring solution, essentially as v^ε was constructed in (4.3).

Lemma 5.5 (Fourier transform of a ring function near $k = 0$). *From a function $V \in C_c^\infty(\mathbb{R}; \mathbb{R})$ we define $w^\varepsilon(x)$, $x \in \mathbb{R}^2$, by*

$$w^\varepsilon(x) := \frac{1}{|x|^{1/2}} V(|x| - \varepsilon^{-2}). \quad (5.22)$$

Then there exist constants $\rho_0, \varepsilon_0, C > 0$ such that

$$\int_{B_\rho(0)} |\hat{w}^\varepsilon(k)|^2 dk \leq C\rho + O(\varepsilon^2) \quad (5.23)$$

for every $\rho \in (0, \rho_0)$ and every $\varepsilon \in (0, \varepsilon_0)$.

Proof. We evaluate the Fourier transformation in polar coordinates and perform a calculation as in (4.18).

$$\begin{aligned} \hat{w}^\varepsilon(k) &= \int_{\mathbb{R}^2} e^{-ik \cdot x} w^\varepsilon(x) dx \\ &= \int_{S^1} \int_0^\infty e^{-ik \cdot qr} w^\varepsilon(rq) r dr dS(q) \\ &= \int_{S^1} \int_0^\infty e^{-ik \cdot qr} V(r - \varepsilon^{-2}) r^{1/2} dr dS(q) \\ &= \int_{S^1} \int_{-\infty}^\infty e^{-ik \cdot q/\varepsilon^2} e^{-ik \cdot qz} V(z) \mathbf{1}_{\{z \geq -\varepsilon^{-2}\}} (\varepsilon^{-2} + z)^{1/2} dz dS(q) \\ &= \int_{S^1} e^{-ik \cdot q/\varepsilon^2} \int_{-\infty}^\infty e^{-ik \cdot qz} V(z) \frac{1}{\varepsilon} dz dS(q) + G_\varepsilon(k) \\ &= e^{-i|k|/\varepsilon^2} \int_{S^1} e^{i|k|/\varepsilon^2} e^{-ik \cdot q/\varepsilon^2} \hat{V}(k \cdot q) \frac{1}{\varepsilon} dS(q) + G_\varepsilon(k). \end{aligned}$$

Since the support of V is bounded, we have $|(\varepsilon^{-2} + z)^{1/2} - \varepsilon^{-1}| = O(\varepsilon)$ for all relevant values of z ; this implies $G_\varepsilon = O(\varepsilon)$. The remaining integral of the last line is treated as in Lemma 5.2: Without loss of generality we consider $k = k_1 e_1$. With $\eta := \varepsilon^2/|k_1|$ we write

$$\begin{aligned} \int_{S^1} e^{i|k|/\varepsilon^2} e^{-ik \cdot q/\varepsilon^2} \hat{V}(k \cdot q) \frac{1}{\varepsilon} dS(q) &= \int_{S^1} e^{i(1-q_1)k_1/\varepsilon^2} \hat{V}(k_1 q_1) \frac{1}{\varepsilon} dS(q) \\ &= \frac{1}{\sqrt{|k_1|}} \int_{S^1} e^{i(1-q_1)/\eta} \hat{V}(k_1 q_1) \frac{1}{\sqrt{\eta}} dS(q) \leq \frac{C}{\sqrt{|k|}}, \end{aligned}$$

where the last step follows with the calculations of Lemma 5.2 by uniform continuity of $\hat{V}$. We can therefore integrate the squared Fourier transform of w^ε ,

$$\begin{aligned} \int_{B_\rho(0)} |\hat{w}^\varepsilon(k)|^2 dk &\leq \int_{B_\rho(0)} \left(\frac{C}{\sqrt{|k|}} \right)^2 dk + O(\varepsilon^2) \\ &= \int_{B_\rho(0)} \frac{C^2}{|k|} dk + O(\varepsilon^2) = 2\pi\rho C^2 + O(\varepsilon^2). \end{aligned}$$

This implies (5.23). □

The estimate (5.23) for $\hat{v}^\varepsilon$ could be the key step in the derivation of (5.21): the k -values with small norm do not contribute much. The difficulty in deriving (5.23) for $\hat{v}^\varepsilon$ is that the cut-off parameter $\rho > 0$ enters in the construction of $\hat{V}^\varepsilon$ and hence in the decay properties of V^ε . For this reason our analysis ends with the result of pointwise convergence in Fourier space.

6 Numerical tests

One space dimension

We test the validity of the linearized KdV equation for initial data $u_0(x) := u_0^\varepsilon(x) := e^{-|x|^2}$, $x \in \mathbb{R}$. The lattice model uses the three point discrete Laplacian, $a_1 = a_{-1} = 1$, $a_0 = -2$, and $a_j = 0$ for every j with $|j| > 1$. All calculations are performed with Matlab.

Lattice model. The initial data u_1^ε are determined by the fact that we demand $\psi_{-,0}^{u,\varepsilon} = 0$, or, equivalently, $\hat{u}_1^\varepsilon(\cdot) = -i\omega_l(\varepsilon)\varepsilon^{-1}\hat{u}_0^\varepsilon(\cdot)$. The calculation of u_1 simplifies considerably in our case: u_0^ε is real and symmetric (invariant under $x \mapsto -x$), which implies that also $\hat{u}_0^\varepsilon$ is real and symmetric (invariant under $\xi \mapsto -\xi$). By its definition, $\hat{u}_1^\varepsilon$ is imaginary and symmetric, hence u_1^ε is imaginary and symmetric. Since we are interested in the real part of the lattice solution, we can perform all calculations with the real parts of the initial data, i.e. with $u_0 = u_0^\varepsilon$ and $u_1 = 0$.

The lattice equations are solved for $\varepsilon = 1/6$ on the truncated domain $x \in (-100, 100)$ with the time interval $t \in [0, t_0]$, $t_0 = 80$. The homogenized wave speed is $c = 1$ such that the main pulses of $u(\cdot, t_0)$ are located near $x = \pm 80$. The part of the solution with $x \in (72, 88)$ is shown in the left part of Figure 1.

Linearized KdV equation. On the other hand, we have solved numerically the linearized KdV equation. The initial values are determined by the definition of $\hat{V}_0^\varepsilon$ in (4.8). We use this definition, the formula for the inverse Fourier transform, and symmetry of $\hat{u}_0^\varepsilon$ to obtain

$$u_0^\varepsilon(x) = \frac{1}{2\pi} \int_{\{\xi > 0\}} e^{i\xi x} \hat{V}_0^\varepsilon(\xi, 1) d\xi + \frac{1}{2\pi} \int_{\{\xi < 0\}} e^{i\xi x} \hat{V}_0^\varepsilon(-\xi, 1) d\xi = 2\text{Re}(V_0^\varepsilon(x, 1)).$$

The result is that we must use the real initial values $V_0 = \frac{1}{2}u_0$ for the linearized KdV equation. The factor $b(q)$ of the linearized KdV equation is determined from the Taylor expansion of the dispersion relation as $b(q) = -\frac{1}{24}$ for $q \in \{\pm 1\}$. The numerical results for $\varepsilon = 1/6$ on the truncated domain $x \in (-8, 8)$ is shown in the right part of Figure 1.

Two space dimensions

The two-dimensional tests are performed for $u_0(x) := u_0^\varepsilon(x) := e^{-|x|^2}$, $x \in \mathbb{R}^2$, and $\varepsilon = 1/6$. The lattice model uses the five point discrete Laplacian, $a_{(1,0)} = a_{(-1,0)} = a_{(0,1)} = a_{(0,-1)} = 1$, $a_0 = -4$, and $a_j = 0$ for every $j \in \mathbb{Z}^2$ with $|j| > 1$. The dispersion relation for this lattice model is

$$\begin{aligned} \omega_l^2(k) &= - \sum_j a_j e^{ik \cdot j} = 4 - e^{ik_1} - e^{-ik_1} - e^{ik_2} - e^{-ik_2} = 4 - 2\cos(k_1) - 2\cos(k_2) \\ &\approx |k_1|^2 + |k_2|^2 - \frac{1}{12}(k_1^4 + k_2^4) = |k|^2 - \frac{1}{12}(k_1^4 + k_2^4), \end{aligned}$$

where we approximated with a Taylor expansion. For the square root we find

$$\omega_l(k) \approx |k| - \frac{1}{24|k|}(k_1^4 + k_2^4).$$

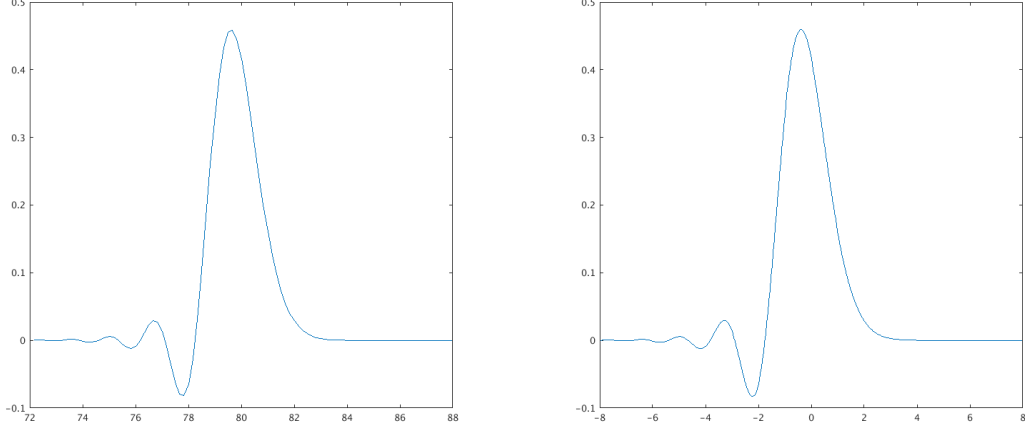


Figure 1: **One-dimensional model.** Left: The solution to the lattice equations with $\varepsilon = 1/6$ at time $t_0 = 80$. We show a zoom into the right-going wave-pulse. Right: Solution to the corresponding linearized KdV equation with $b = -1/24$ and initial data $V_0(x) = \frac{1}{2}u_0(x) = \frac{1}{2}e^{-|x|^2}$ at time $t_0\varepsilon^2$.

This determines c and b through (4.2). We find $c = 1$ and, for the two directions $q = e_1$ and $q = e_2$, the values $b(e_1) = b(e_2) = -1/24$.

We next calculate an approximation of $\hat{u}_0^\varepsilon$ by replacing the discrete Fourier transform with the continuous transform. Our choice of constants in the definition of the Fourier transform suggests to use

$$\hat{u}_0(\xi) = \int_{\mathbb{R}^2} e^{-|x|^2} e^{-ix \cdot \xi} dx = \pi e^{-|\xi|^2/4}. \quad (6.1)$$

Lattice model. The initial data u_1^ε are determined by the fact that we demand $\psi_{-,0}^{u,\varepsilon} = 0$, or, equivalently, $\hat{u}_1^\varepsilon(\cdot) = -i\omega_l(\varepsilon)\varepsilon^{-1}\hat{u}_0^\varepsilon(\cdot)$. As in the one-dimensional case, the Fourier transform of u_1^ε is symmetric and purely imaginary, hence the same holds for u_1^ε . Since we calculate the solution to the lattice equations for real functions u_0 and u_1 , we can use the homogeneous initial values $u_1 = 0$.

We present here two numerical solutions for the two-dimensional lattice equations. Figure 2 shows the result for $\varepsilon = 1/6$, $t \in [0, t_0]$ with $t_0 = 20$, $x \in (-25, 25)^2$, calculated with an explicit solver for the original equations (1.1) with time step size $5 \cdot 10^{-5}$. Figure 3 shows a detail from a calculation on a larger domain, $\varepsilon = 1/6$, $t_0 = 80$, $x \in (-120, 120)^2$; here, we solved the representation formula (2.7) with periodic boundary conditions. The homogenized wave speed is $c = 1$ such that the main pulse of $u(\cdot, t_0)$ is near $|x| = t_0$.

Linearized KdV equation. We use (6.1) to evaluate $\hat{V}_0$ from (5.1). For every $q \in S^1$ and $\xi > 0$ we find

$$\hat{V}_0(\xi; q) = (\sqrt{\pi}(1+i))^{-1} \sqrt{\xi} \hat{u}_0(\xi q) = \frac{1-i}{2\sqrt{\pi}} \sqrt{\xi} \hat{u}_0(\xi q) = \frac{1}{2}(1-i) \sqrt{\pi\xi} e^{-|\xi|^2/4}.$$

The real initial data for the linearized KdV equation are obtained by the one-dimensional inverse Fourier transform of the above function for arbitrary q . We

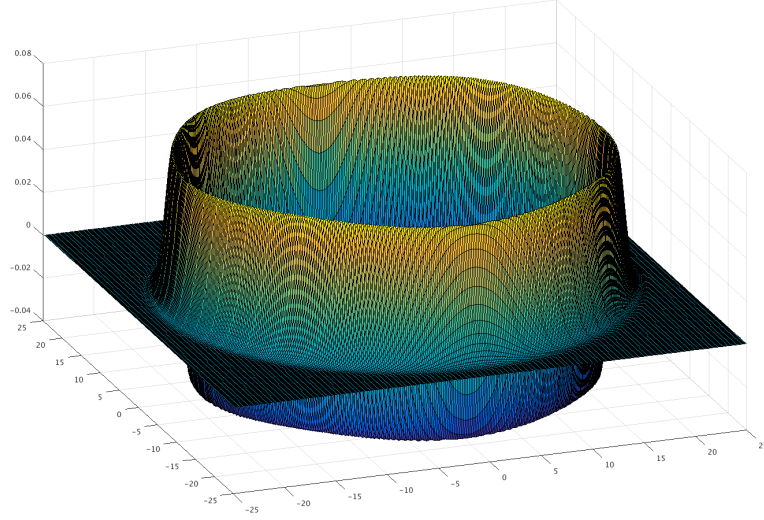


Figure 2: The two-dimensional solution $x \mapsto u(x, t_0)$ of the lattice equations for $\varepsilon = 1/6$, $x \in (-25, 25)^2$, $t_0 = 20$, $u_0(x) = e^{-|x|^2}$. We see that the solution takes the form of a ring.

obtain

$$V_0(x; q) = \frac{1}{4\sqrt{\pi}} \operatorname{Re} \int_{\{\xi > 0\}} (1 - i) \sqrt{\xi} e^{-|\xi|^2/4} e^{i\xi x} d\xi.$$

The left part of Figure 5 shows the numerically determined initial values $x \mapsto V_0(x; q)$ for $q = e_1$. The right part of Figure 5 shows the numerically obtained solution $V(\cdot, \tau_0; e_1)$ for $\tau_0 = t_0 \varepsilon^2$. The evolution equation is solved for $q = e_1$, using $b(e_1) = -1/24$ as coefficient of the third order term and time step size $1 \cdot 10^{-7}$.

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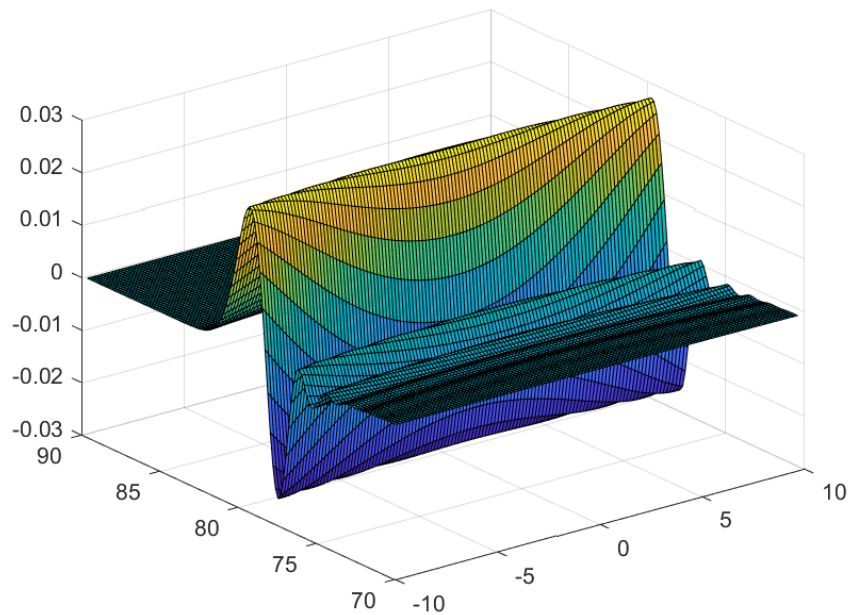


Figure 3: A detail of the two-dimensional solution $x \mapsto u(x, t_0)$ of the lattice equations for $\varepsilon = 1/6$, $t_0 = 80$, solved for $x \in (-120, 120)^2$ and shown for $x \in (70, 90) \times (-10, 10)$, $u_0(x) = e^{-|x|^2}$. We show a zoom into the front to make the profile visible.

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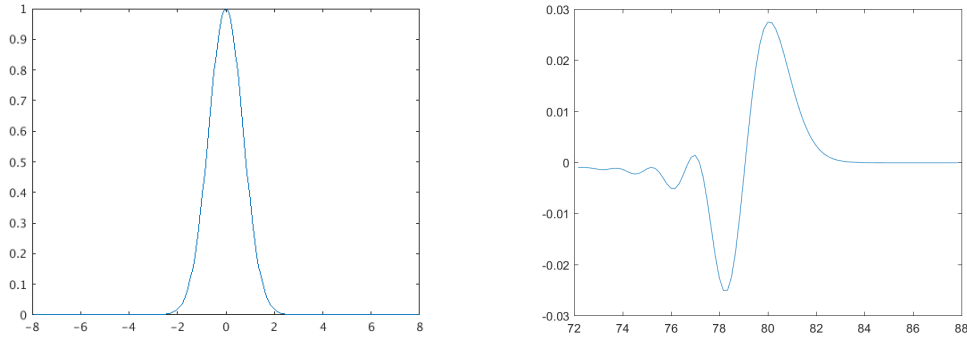


Figure 4: **Lattice solution.** Two-dimensional lattice model for $\varepsilon = 1/6$, solutions are shown along the line $(x_1, 0)$. Left: Initial values u_0 , plotted is the function $x_1 \mapsto u_0((x_1, 0))$. Right: The solution of the lattice model at time $t_0 = 80$, plotted is $x_1 \mapsto u((x_1, 0), t_0)$. The data are the same as for Figure 3.

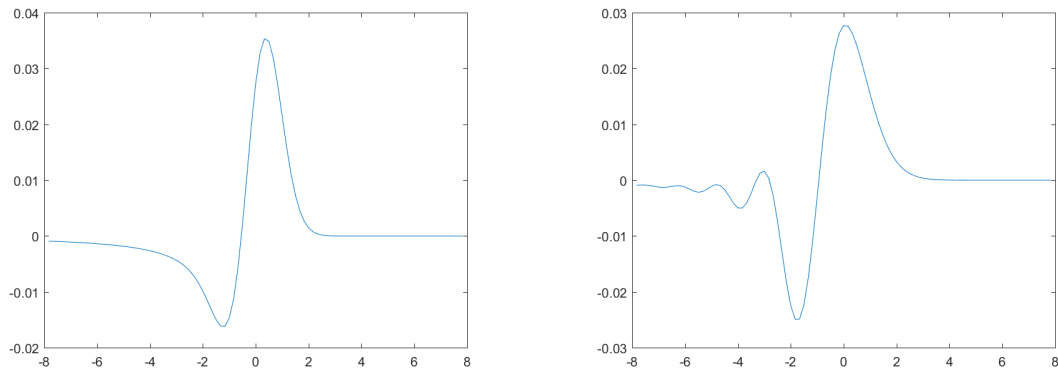


Figure 5: **KdV-solution.** Two-dimensional model approximated with a one-dimensional KdV-equation. Left: The initial values V_0 , obtained from the Fourier transform of u_0 . We plot $V_0/\sqrt{r_0}$ for the radius $r_0 = ct_0 = t_0 = 80$ in order to match the reconstruction formula (4.3). Right: The solution of the linearized KdV equation at time $t = t_0\varepsilon^2$, also divided by $\sqrt{r_0}$. The KdV-solution matches almost perfectly the two-dimensional lattice solution, see the right part of Figure 4.

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