A special perspective on homogenization — with an application to plasticity

Trends on Applications of Mathematics to Mechanics

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Mathematical analysis of plasticity

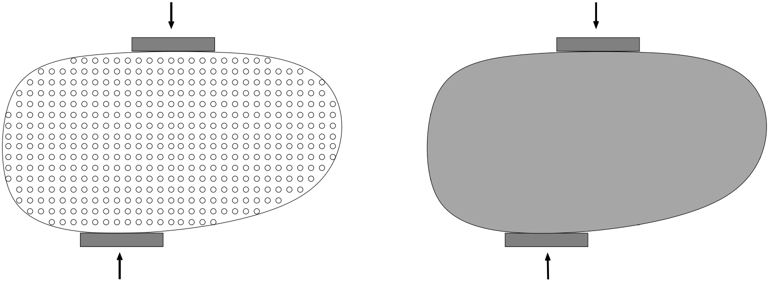
What can be done mathematically in "plasticity"?

(a personal selection)

1. Analysis of crystal plasticity, derivation of macro-equations
2. Nonlinear theories
3. **Homogenization** (starting from a continuum model)
    - H.-D. Alber(2000)
    - A. Visintin (2005, 2006)
    - A. Mielke & A.M. Timofte / U. Stefanelli / T. Roubicek (2007, 2008, 2013)
    - (H.-D. Alber &) S. Nesenenko (2007, 2009)
    - B.S. & M. Veneroni (2011 & 2014)
    - M. Heida & B.S. (2015 & submitted)

What is homogenization?

A heterogeneous material occupies a set $\Omega \subset \mathbb{R}^3$. How does it deform under applied forces?

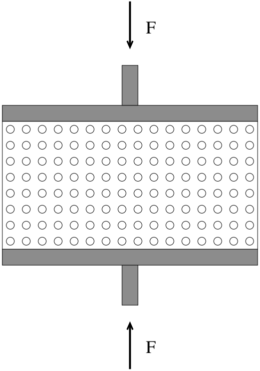


Aim of homogenization

In good approximation:
the **heterogeneous** material (left)
behaves like the **homogeneous** *effective* material (right).

What can we measure?

We can make experiments with a heterogeneous test-volume. We deform the specimen (shear and compression) and measure the response



Measurements yield:

- the averaged deformation $\bar{e} \in \mathbb{R}^{n \times n}$ **leads to**
- the averaged force $\bar{\sigma} \in \mathbb{R}^{n \times n}$

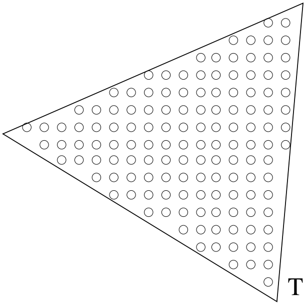
Result

We determined the tensor $\mathbf{A}$ in the law

$$\bar{\sigma} = \mathbf{A} \cdot \bar{e}$$

Effective tensor from deformation experiment

How can we understand the effect of coefficients a_ε?



1. Consider a simplex $\mathcal{T} \subset \mathbb{R}^n$ and a deformation tensor $\xi \in \mathbb{R}^{n\times n}$
2. Impose affine boundary data $U_\xi : \mathbb{R}^n \to \mathbb{R}^n$, $U_\xi(x) := \xi \cdot x$
3. Solve the problem $-\nabla \cdot (a_\varepsilon \nabla u^\varepsilon) = 0$ on $\mathcal{T}$ with $u^\varepsilon = U_\xi$ on $\partial\mathcal{T}$

Assumption: The averaged stress is

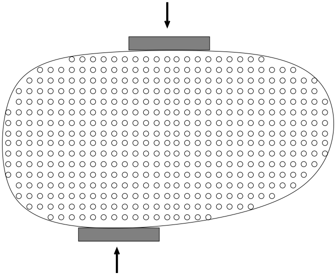
$$\lim_{\varepsilon\to 0} \frac{1}{|\mathcal{T}|} \int_{\mathcal{T}} a_\varepsilon \nabla u^\varepsilon = A \cdot \xi$$

Under strain ξ, the average stress in the test-volume is $A \cdot \xi$

Homogenization in more mathematical terms

Heterogeneous material in $\Omega \subset \mathbb{R}^3$. The deformation is

$$u^\varepsilon : \Omega \to \mathbb{R}^3$$



$$\sigma^\varepsilon = a_\varepsilon \nabla u^\varepsilon$$

$$-\nabla \cdot \sigma^\varepsilon = f$$

Effective material description:

$$u^\varepsilon \approx u$$

- deformation $u : \Omega \to \mathbb{R}^3$ with
- stress tensor $\sigma : \Omega \to \mathbb{R}^{3\times 3}$

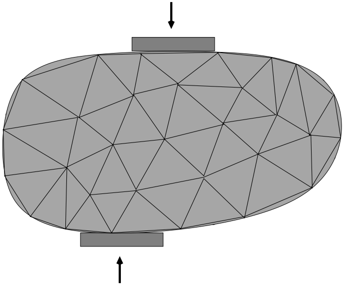
The aim of homogenization

The effective law in Ω is

$$-\nabla \cdot \sigma = f \quad \text{with} \quad \sigma = \mathbf{A} \cdot \nabla u$$

Coefficients that allow averaging

In this spirit, non-periodic problems: Needle problem approach!



Assumption: The *single triangle* reacts on an affine deformation U_ξ (with $\nabla U_\xi \equiv \xi$) with the force $A \cdot \xi$

Definition

The coefficients a_ε **allow averaging** of the constitutive relation with $A \in \mathbb{R}^{n \times n}$ if, for simplices $\mathcal{T}$, solutions of

$$-\nabla \cdot (a_\varepsilon \nabla u^\varepsilon) = 0 \qquad \text{in } \mathcal{T}$$
$$u^\varepsilon = U_\xi \qquad \text{on } \partial\mathcal{T},$$

with affine boundary data U_ξ satisfy

$$\lim_{\varepsilon \to 0} \frac{1}{|\mathcal{T}|} \int_{\mathcal{T}} a_\varepsilon \nabla u^\varepsilon = A \cdot \xi$$

The needle-problem homogenization method

Theorem (B.S. and M. Veneroni, 2011)

Assume that the coefficients a_ε **allow averaging** *of the constitutive relation with $A \in \mathbb{R}^{n\times n}$.*

Then: *Solutions u^ε of $-\nabla \cdot (a_\varepsilon \nabla u^\varepsilon) = f$ on domains Ω converge to solutions u^* of $-\nabla \cdot (A \cdot \nabla u^*) = f$.*

The proof uses:

- triangulations
- approximate solutions
- adapted grids and a new div-curl Lemma

$$
\begin{array}{ccc}
u^\varepsilon & \overset{\varepsilon, h \to 0}{\longleftrightarrow} & u_h^\varepsilon \\
 & & \Big\downarrow \varepsilon \to 0 \\
u^* & \overset{h \to 0}{\longleftarrow} & u_h
\end{array}
$$

Definition of the needle problem

Approximate solutions: defined as the solutions u_h^ε of the

Definition (Needle problem)

Given a Lipschitz domain $\Omega \subset \mathbb{R}^n$, a triangulation $\mathscr{T}_h$ of $\Omega_h \subset \Omega$, and piecewise affine boundary data ψ. Function space:

$$\mathcal{N}_h := \left\{ \phi \in H_0^1(\Omega) \ : \ \phi|_{\partial T_k} \text{ is affine for all } T_k \in \mathscr{T}_h, \ \phi \equiv 0 \text{ on } \Omega \setminus \Omega_h \right\}$$

Given g_h, the *needle problem* is to find $u_h^\varepsilon \in \psi + \mathcal{N}_h$ such that

$$\int_\Omega a_\varepsilon \nabla u_h^\varepsilon \cdot \nabla \phi = \int_{\Gamma_h} g_h \phi \qquad \forall \phi \in \mathcal{N}_h$$

Here: f is replaced by g_h such that

$$\int_{\bigcup \partial T_k} g_h \cdot \varphi = \int_\Omega f \cdot \varphi \quad \text{for piecewise affine test functions } \varphi$$

Proof (homogenization of the elastic problem)

Main part of the homogenization proof:

Lemma (Comparison of u_h^ε and u^ε)

$a_\varepsilon \in L^\infty(\Omega;\mathbb{R}^{n\times n})$ elliptic, ψ boundary values, $u^\varepsilon \in H^1(\Omega)$ solutions of original problem, $u_h^\varepsilon \in \psi + \mathcal{N}_h$ solutions to needle problem. If $\mathscr{T}_h$ are adapted grids for $(u^\varepsilon)_\varepsilon$, then

$$\lim_{h\to 0}\lim_{\varepsilon\to 0} \|u_h^\varepsilon - u^\varepsilon\|_{H^1(\Omega)} = 0.$$

Idea of proof:

1. use $(u^\varepsilon - u_h^\varepsilon)$ as a test-function in the original problem
2. use $(u^\varepsilon - u_h^\varepsilon)$ as a test-function in the needle problem
3. difference yields estimate for $\|u^\varepsilon - u_h^\varepsilon\|_{H^1(T)}^2$
4. show smallness of error terms for $\varepsilon \to 0$ and then $h \to 0$

the last step uses a div-curl lemma on each triangle

Adapted grids

Theorem (Adapted grids and div-curl lemma)

Let $\Omega \subset \mathbb{R}^n$, $n = 2$ or $n = 3$ be a bounded Lipschitz domain, $(u^\varepsilon)_\varepsilon$ a bounded sequence in $H^1(\Omega)$.

1. *To arbitrary $h > 0$ there exists $\Omega_h \subset \Omega$ and an adapted triangulation $\mathscr{T}_h$ of Ω_h for $(u^\varepsilon)_\varepsilon$.*
2. *Let $(u^\varepsilon)_\varepsilon$ be a sequence with $u^\varepsilon \rightharpoonup u$ weakly in $H^1(\Omega)$ and let $\mathscr{T}_h$ be an adapted grid for $(u^\varepsilon)_\varepsilon$. Let $(q^\varepsilon)_\varepsilon$ be a sequence in $L^2(\Omega, \mathbb{R}^n)$ satisfying*

$$q^\varepsilon \rightharpoonup q \text{ weak in } L^2(\Omega),$$

$$f^\varepsilon := \nabla \cdot q^\varepsilon \to f \text{ strong in } H^{-1}(T), \quad \text{for all } T \in \mathscr{T}_h.$$

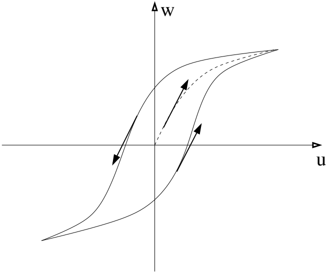
Then

$$\lim_{\varepsilon \to 0} \int_{\Omega_h} q^\varepsilon \cdot \nabla u^\varepsilon \, dx = \int_{\Omega_h} q \cdot \nabla u \, dx.$$

Idea: boundary values of u^ε are bounded in H^1, hence compact in $H^{1/2}$

Plasticity as a system with hysteresis

Important in plasticity: **Hysteresis!**



The stress $\sigma(x, t)$ depends not only on the current deformation tensor $\nabla u(x, t)$, but also on its history:

$$\sigma(x, t) = \mathcal{F}\left(\{\nabla u(x, s) | s \in [0, t]\}\right)$$

Plasticity (Prandtl-Reuss model)

Reference domain of material: Ω, time interval: $(0, T)$

Variables

displacement	u	$: \Omega \times (0,T) \to \mathbb{R}^n$
strain	$\nabla^s u$	$: \Omega \times (0,T) \to \mathbb{R}^{n\times n}$
stress	σ	$: \Omega \times (0,T) \to \mathbb{R}^{n\times n}$

with $\nabla^s u = \frac{1}{2}(\nabla u + (\nabla u)^T)$

Equations

Conservation of momentum: $-\nabla \cdot \sigma = f$

Additive strain decomposition: $\nabla^s u = \underbrace{e}_{\text{elastic strain}} + \underbrace{p}_{\text{plastic strain}}$

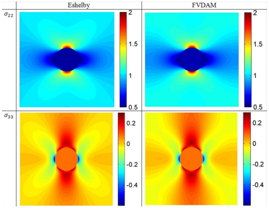
Hooke's law: $C\sigma = e$

Flow rule with kinematic hardening: $\partial_t p \in \partial\Psi(\sigma - Bp)$

Given: force f, elasticity tensor C, hardening tensor B, potential Ψ

What can we expect from homogenization?

The periodicity cell remembers its *deformation history*



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We cannot expect to find a map

$$\mathbb{R}^{n\times n} \ni \bar{e} \mapsto \bar{\sigma} \in \mathbb{R}^{n\times n}$$

(current local strain)
$\mapsto$ (current local stress)

Definition (Averaging property in a system with hysteresis)

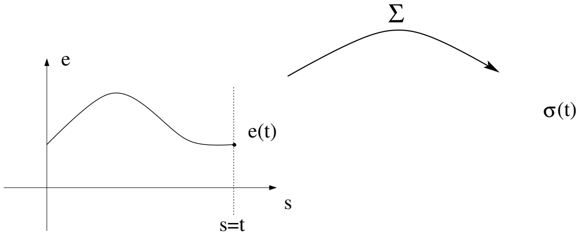
The ε-system allows averaging if there exists an operator

$$\Sigma \ : \ H^1(0,T;\mathbb{R}^{n\times n}_s) \to H^1(0,T;\mathbb{R}^{n\times n}_s)$$

such that: For a simplex $\mathcal{T} \subset \mathbb{R}^n$, $\xi \in H^1(0,T;\mathbb{R}^{n\times n}_s)$ and solutions u^ε, e^ε, p^ε, σ^ε to the ε-problem on $\mathcal{T}$ with $f = 0$ and $u^\varepsilon(t)|_{\partial\mathcal{T}} = \xi(t)\cdot x$ holds:

$$\fint_{\mathcal{T}} \sigma^\varepsilon(t) \to \Sigma(\xi)(t)$$

An averaging property



An evolution of strain is mapped to a stress

Theorem (Abstract homogenization, M. Heida and B.S.)

Given Ω and f, let the coefficients allow averaging with a lower semi-continuous stress operator Σ. Then the effective problem

$$-\nabla \cdot \Sigma(\nabla^s u) = f \qquad \text{in } \Omega \times (0,T)$$

has a solution u. As $\varepsilon \to 0$, there holds $u^\varepsilon \rightharpoonup u$ in $H^1(0,T;H^1(\Omega))$.

$$\Sigma(\nabla^s u)(x,t) = \Sigma(\nabla^s u(x,.))(t)$$

Needle problem for plasticity

Discretization of $\Omega_h \subset \Omega$ with grid $\mathbb{T}_h = \{\mathcal{T}_k\}_{k\in\Lambda_h}$
$\mathcal{N}_h := \{\phi \in H^1_0(\Omega)\ :\ \phi|_{\partial\mathcal{T}_k} \text{ is affine } \forall\, k \in \Lambda_h,\ \phi \equiv 0 \text{ on } \Omega \setminus \Omega_h\}$
As before: f replaced by g_h such that

$$\int_{\bigcup \partial\mathcal{T}_k} g_h \cdot \varphi = \int_\Omega f \cdot \varphi \quad \text{for piecewise affine } \varphi$$

Definition (Needle problem in plasticity)

Find $u_h^\varepsilon \in H^1(0,T;\mathcal{N}_h)$, $e_h^\varepsilon, p_h^\varepsilon, \sigma_h^\varepsilon \in H^1(0,T;L^2(\Omega_h;\mathbb{R}^{n\times n}_s))$:

$$\int_0^T \int_{\Omega_h} \sigma_h^\varepsilon : \nabla\varphi = \int_0^T \int_{\bigcup \partial\mathcal{T}_k} g_h \cdot \varphi \qquad \forall \varphi \in L^2(0,T;\mathcal{N}_h)\,,$$

and almost everywhere in Ω_h holds

$$\nabla^s u_h^\varepsilon = e_h^\varepsilon + p_h^\varepsilon\ , \qquad C_\varepsilon \sigma_h^\varepsilon = e_h^\varepsilon\ , \qquad \partial_t p_h^\varepsilon \in \partial\Psi_\varepsilon\left(\sigma_h^\varepsilon - B_\varepsilon p_h^\varepsilon\right)$$

Homogenization proof

u_h^ε = solution of the auxiliary problem, "needle problem"

1. $\lim_{h\to 0}\lim_{\varepsilon\to 0}\|u_h^\varepsilon - u^\varepsilon\|_{L^\infty(0,T;H^1(\Omega))} \to 0$
   - Testing with $\partial_t\left(u_h^\varepsilon - u^\varepsilon\right)$
2. $u_h^\varepsilon \to u_h$ as $\varepsilon \to 0$ where u_h is piecewise affine and solves

$$\int_0^T \int_\Omega \Sigma(\nabla^s u_h) : \nabla\varphi = \int_0^T \int_\Gamma g_h \cdot \varphi \qquad \forall \varphi \in L^2(0,T;Y_h)$$

$Y_h := \left\{\phi \in H_0^1(\Omega) \;:\; \phi|_{\mathcal{T}_k} \text{ is affine } \forall\, k \in \Lambda_h,\; \phi \equiv 0 \text{ on } \Omega \setminus \Omega_h\right\}$

   - Follows from averaging property
3. $u_h \to u$ as $h \to 0$ where u solves the effective problem
   - Standard finite element calculation

Stochastic homogenization

Standard setting of stochastic homogenization: Probability space $(\Omega_{\mathcal{P}}, \Sigma_\Omega, \mathcal{P})$, ergodic dynamical system $(\tau_x)_{x\in\mathbb{R}^n}$, the **random coefficients** are

$$C_\varepsilon(x) := C(\tau_{\frac{x}{\varepsilon}}\omega)\,, \quad B_\varepsilon(x) := B(\tau_{\frac{x}{\varepsilon}}\omega)\,, \quad \Psi_\varepsilon(\sigma; x) := \Psi(\sigma; \tau_{\frac{x}{\varepsilon}}\omega)\,.$$

Theorem (Stochastic homogenization, M. Heida and B.S.)

For $\Sigma\;:\;H^1(0,T;\mathbb{R}^{n\times n}_s) \to H^1(0,T;\mathbb{R}^{n\times n}_s)$ from stochastic averaging, the abstract homogenization theorem can be applied:

$$-\nabla\cdot\Sigma(\nabla^s u) = f \qquad \text{in } \Omega\times(0,T)$$

has a solution $u:\Omega\times(0,T)\to\mathbb{R}^n$ and there holds $u^\varepsilon \rightharpoonup u$ as $\varepsilon\to 0$.

Σ is given through a cell-problem: Given $\xi : [0,T] \to \mathbb{R}^{n\times n}$, solve

$$\partial_t p(t,\omega) \in \partial\Psi\left(z(t,\omega) - B(\omega)\,p(t,\omega)\,;\,\omega\right)\,, \quad C\,z = \xi + v^s - p\,,$$

with $z(t) \in L^2_{sol}(\Omega_{\mathcal{P}})$, $v(t) \in L^2_{pot}(\Omega_{\mathcal{P}})$, $\qquad \Sigma(\xi) := \int_{\Omega_{\mathcal{P}}} z\,d\mathcal{P}$

Conclusions

- We presented a new view-point on homogenization
  - needle-problem approach
  - homogenization as two-step procedure
- The method allows to
  - recover known results in elasticity
  - obtain new results in plasticity: **stochastic homogenization**
- Important tool: Adapted grids on which a div-curl lemma holds

Thank you!