Homogenization of domain perforations and sound absorption

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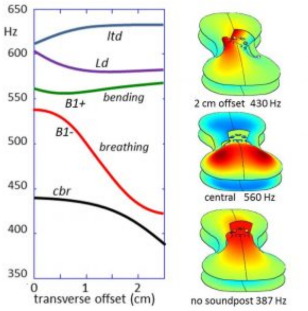
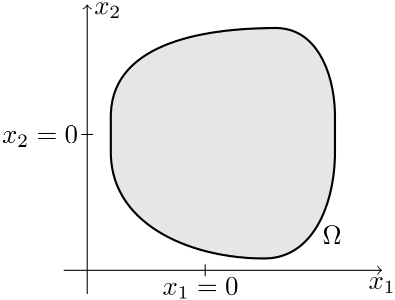
Helmholtz equation



Graphic taken from: Acoustics Today



A domain $\Omega \subset \mathbb{R}^n$

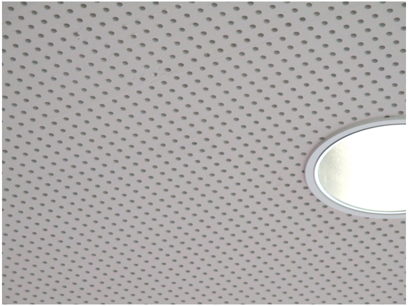
Sound is described by the wave equation $\partial_t^2 p = \Delta p$
Time-harmonic ansatz $p(x,t) = u(x)e^{i\omega t}$ leads to

Helmholtz equation

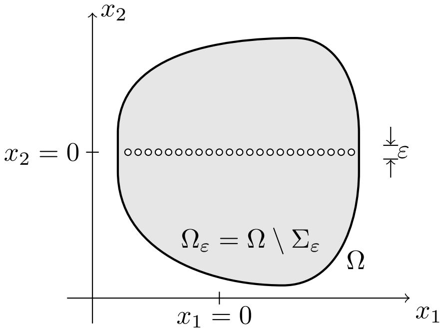
$$-\Delta u = \omega^2 u + f \qquad \text{in } \Omega$$

$f \in L^2(\Omega)$ is a prescribed source

Sound absorbers



$\longrightarrow$ homogenization of perforations



Puzzling fact: Wavelength of sound $\approx$ 1m, holes in the wall $\approx$ 1cm

Dirichlet condition on $\partial\Omega$, Neumann condition along inclusions

Helmholtz equation

$$-\Delta u^\varepsilon = \omega^2 u^\varepsilon + f \qquad \text{in } \Omega_\varepsilon$$

What is the effect of the perforation?

Notation

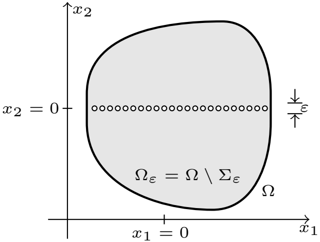
Inclusions: Index $k \in \mathbb{Z}^{d-1}$. The single inclusion is

$$\Sigma_k^\varepsilon := \varepsilon\,(\Sigma + (k,0)) \quad \text{for} \quad k \in \mathbb{Z}^{d-1}$$

Number of inclusions $\sim \varepsilon^{-(d-1)}$

Perforated domain:

$$\Sigma_\varepsilon := \bigcup_{k \in I_\varepsilon} \Sigma_k^\varepsilon \qquad \Omega_\varepsilon := \Omega \setminus \bar{\Sigma}_\varepsilon$$

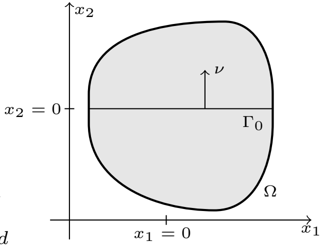


Limit geometry: The perforation Σ_ε is located along the submanifold

$$\Gamma_0 := \left(\mathbb{R}^{d-1} \times \{0\}\right) \cap \Omega$$

Normal vectors: $n = n_\varepsilon(x)$ the outer normal of Ω_ε

The interface has the upward pointing normal $\nu = e_d$



A surprising observation

Extension: $\mathcal{P}_\varepsilon : L^2(\Omega_\varepsilon) \to L^2(\Omega)$ maps a function to its trivial extension
We always assume that ω^2 is not a Dirichlet eigenvalue of $-\Delta$ on Ω

- Multiply the Helmholtz equation $-\Delta u^\varepsilon = \omega^2 u^\varepsilon + f$ with u^ε
  Integrate by parts and use Poincaré $\longrightarrow$ $\|u^\varepsilon\|_{H^1(\Omega_\varepsilon)} \leq C$
- $\mathcal{P}_\varepsilon u^\varepsilon \to u$ strongly in $L^2(\Omega)$ with $u \in H^1(\Omega)$

Result: The limit function u is the solution of

$$-\Delta u = \omega^2 u + f \qquad \text{in } \Omega$$

$\longrightarrow$ **The perforation has no effect!** (at leading order)

Error estimate:

$$\|u - \mathcal{P}_\varepsilon u^\varepsilon\|_{L^2(\Omega)} + \|\nabla u - \mathcal{P}_\varepsilon \nabla u^\varepsilon\|_{L^2(\Omega)} \leq C\varepsilon^{1/2}$$

C. Dörlemann, M. Heida, and B. Schweizer. Transmission conditions for the Helmholtz-equation in perforated domains. *Vietnam J. Math.*, 45(1-2):241–253, 2017

Interesting limits occur at first order

u^ε: solution on Ω_ε $\quad$ u: solution of the limit equation on Ω

Define the corrector

$$v^\varepsilon := \frac{u^\varepsilon - u}{\varepsilon}$$

Assume $v^\varepsilon \to v$. What are the equations for v?

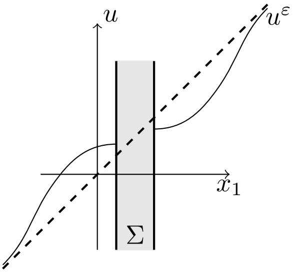
Orders of magnitude

- ∇u is smooth, order $O(1)$ around inclusion (dashed line)
- $n \cdot \nabla v^\varepsilon = -\frac{1}{\varepsilon} n \cdot \nabla u$ of order $O(\varepsilon^{-1})$
- v^ε has variations $O(1)$

Functions spaces

bad: $\|\nabla v^\varepsilon\|_{L^2(\Omega_\varepsilon)} \to \infty$ expected

good: $\|\nabla v^\varepsilon\|_{L^1(\Omega_\varepsilon)} \leq C$ possible



u and u^ε near an obstacle

divide et impera!

Assumption. With $C > 0$ independent of ε holds

$$\|v^\varepsilon\|_{W^{1,1}(\Omega_\varepsilon)} \leq C$$

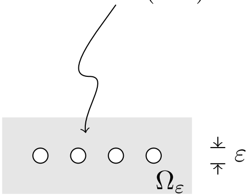
Two questions

1. What are the equations for v?
2. Why should v^ε satisfy the $W^{1,1}$-bound?

The $W^{1,1}$-bound implies for some $q > 1$, $v \in L^1(\Omega)$:

- $\mathcal{P}_\varepsilon v^\varepsilon \overset{*}{\rightharpoonup} v \, d\mathcal{L}^d$ weak-$*$ as measures
- $\mathcal{P}_\varepsilon \nabla v^\varepsilon \overset{*}{\rightharpoonup} \nabla v + \mu$ for some measure μ with $\mathrm{supp}(\mu) \subset \Gamma_0$
- $v \in L^q_{\mathrm{loc}}(\Omega)$ and $\mathcal{P}_\varepsilon v^\varepsilon \to v$ in $L^1_{\mathrm{loc}}(\Omega)$
- $v \in W^{1,1}(\Omega \setminus \Gamma_0)$

$$v^\varepsilon = O(1)$$
$$\nabla v^\varepsilon = O(\varepsilon^{-1})$$



Orders of magnitude near an obstacle

The main result

Theorem (Effective system for the corrector)

Let u^ε *and* u *be as above,* $v^\varepsilon := \frac{u^\varepsilon - u}{\varepsilon}$
Assume the $W^{1,1}$*-bound, and* $\mathcal{P}_\varepsilon v^\varepsilon \to v$
Then $v \in W^{1,1}(\Omega \setminus \Gamma_0)$ *is the unique solution of*

$$\begin{aligned} -\Delta v &= \omega^2 v && \text{in } \Omega \setminus \Gamma_0 \\ [v] &= J \cdot \nabla u && \text{on } \Gamma_0 \\ [\partial_\nu v] &= \nabla \cdot (G \nabla u) && \text{on } \Gamma_0 \end{aligned}$$

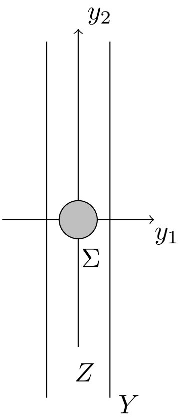
The matrices $G \in R^{d\times d}$ and $J \in \mathbb{R}^d$ are given by cell problems

B. Delourme, H. Haddar, and P. Joly. Approximate models for wave propagation across thin periodic interfaces. *J. Math. Pures Appl. (9)*, 98(1):28–71, 2012.

B. Schweizer. Effective Helmholtz problem in a domain with a Neumann sieve perforation. *J. Math. Pures Appl. 142:1-22, 2020.*

Cell problems

$$Y := \left(-\frac{1}{2}, \frac{1}{2}\right)_{\text{per}}^{d-1} \times \mathbb{R} \qquad\qquad Z := Y \setminus \Sigma$$



Definition: Cell problem

Given $\xi \in \mathbb{R}^d$, seek $w \in H^1_{\text{loc}}(Z)$ such that

$$\begin{aligned} -\Delta w &= 0 && \text{in } Z \\ \partial_n w &= n \cdot \xi && \text{on } \partial\Sigma \end{aligned}$$

$n : \partial\Sigma \to \mathbb{R}^d$ is the exterior normal of Z

"Gradient":

$$G\,\xi := \int_Z \nabla w \in \mathbb{R}^d$$

"Jump":

$$J \cdot \xi := -\lim_{\zeta \to \infty} \int_{\{y_d = \zeta\}} w + \lim_{\zeta \to -\infty} \int_{\{y_d = \zeta\}} w \in \mathbb{R}$$

Idea of the proof: Elementary unfolding

Let $\varphi \in C_c^\infty(\Omega)$ be arbitrary. Consider $V_\varphi^\varepsilon : Z \to \mathbb{R}$,

$$V_\varphi^\varepsilon(y) := \frac{1}{|I_\varepsilon|} \sum_{k \in I_\varepsilon} v^\varepsilon(\varepsilon(k+y))\, \varphi(\varepsilon(k+y))$$

Derive estimates for V_φ^ε using $\|\nabla v^\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq C_0\, \varepsilon^{-1/2}$ and conclude

$$V_{\varphi,0}^\varepsilon \rightharpoonup w \text{ in } \dot{H}^1(Z)$$

as $\varepsilon \to 0$. Here w is the cell-problem solution for

$$\xi := -\frac{1}{|\Gamma_0|} \int_{\Gamma_0} \nabla u\, \varphi \in \mathbb{R}^d$$

Furthermore, boundary integrals also converge:

$$e_j \cdot \int_{\partial\Sigma_\varepsilon} n\, v^\varepsilon\, \varphi \to |\Gamma_0| e_j \cdot \int_{\partial\Sigma} n\, w$$

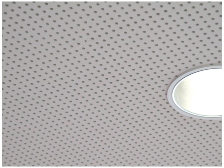
Conclude that, in the convergence

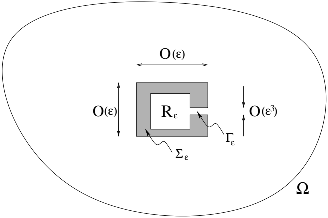
$$\mathcal{P}_\varepsilon \nabla v^\varepsilon \overset{*}{\rightharpoonup} \nabla v + \mu\,,$$

there holds $\mu = -G \nabla u\, \mathcal{H}^{d-1}\lfloor_{\Gamma_0}$

The Helmholtz resonator

Recall the puzzling fact: Wavelength of sound $\approx$ 1m, holes $\approx$ 1cm





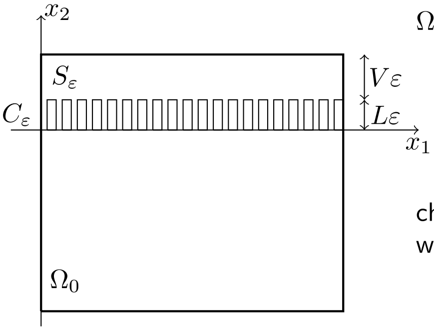
The operator $(-\Delta)^{-1}$ in $L^2(\Omega \setminus \Sigma_\varepsilon)$ has an eigenvalue μ_ε with

$$\mu_\varepsilon \to \mu_0 = \frac{LV}{A} = \lim_{\varepsilon \to 0} \frac{L_\varepsilon |R_\varepsilon|}{|\Gamma_\varepsilon|}$$

L: length of the channel, V: volume of the resonator, A: opening area

Result: For fixed frequency, the resonator can be arbitrarily small

The three-scale geometry



Ω_ε: the complex domain, union of

- limit domain Ω_0 (below x_1-axis)
- channels C_ε
- the strip S_ε above the channels

channels distributed with periodicity ε, width is $\alpha\varepsilon^3$

We investigate the Helmholtz problem in two dimensions ($n = 2$)

$$-\Delta u^\varepsilon - \omega^2 u^\varepsilon = f \qquad \text{in } \Omega_\varepsilon$$
$$\partial_n u^\varepsilon = 0 \qquad \text{on } \partial\Omega_\varepsilon$$

P. Donato, A. Lamacz, B. Schweizer. Sound absorption by perforated walls along boundaries. *Applicable Analysis*, 2020

Trivial limit

Trivial limit problem

$$-\Delta u - \omega^2 u = f \qquad \text{in } \Omega_0$$

Quantities of interest:

$$w^\varepsilon : \Omega_0 \to \mathbb{R}, \qquad w^\varepsilon := \frac{u^\varepsilon - u}{\varepsilon}$$

and "u^ε behind the perforated wall", $I = (0,1)$: horizontal cross-section

$$v^\varepsilon : I \to \mathbb{R}, \qquad x_1 \mapsto \frac{1}{\varepsilon V} \int_{\varepsilon L}^{\varepsilon(L+V)} u^\varepsilon(x_1, x_2)\, dx_2$$

Theorem (System for limits $v^\varepsilon \to v$ and $w^\varepsilon \to w$)

$$-\Delta w - \omega^2 w = 0 \qquad \textit{in } \Omega_0$$

$$\partial_n w = V\,(\partial_1^2 + \omega^2) v \qquad \textit{on } \Gamma_0$$

and $\partial_n w = 0$ on $\partial\Omega_0 \setminus \Gamma_0$. The quantity v solves

$$\left(-\partial_1^2 + \left(\frac{\alpha}{LV} - \omega^2\right)\right) v = \frac{\alpha}{LV} u|_{\Gamma_0}$$

A flux quantity

New quantity: Vertical flux j^ε together with its limit j_*

$$j^\varepsilon(x) := \frac{1}{L\varepsilon^2}\,\partial_2 u^\varepsilon(x)\,\mathbf{1}_{C_\varepsilon}(x)\,,$$

where $\mathbf{1}_{C_\varepsilon}$ is the characteristic function of the channels
Up to a subsequence $\varepsilon \to 0$: There exists a measure $j_* \in \mathcal{M}(\mathbb{R}^2)$ with $\mathrm{supp}(j_*) \subset \bar{\Gamma}_0$ and, as measures,

$$j^\varepsilon \overset{*}{\rightharpoonup} j_*\,, \qquad j_*(x) = j(x_1)\,\mathcal{H}^1|_{\Gamma_0}$$

Why this scaling?

- u^ε on the upper end of Ω_0 and v^ε differ by $O(1)$
- Hence $\partial_2 u^\varepsilon$ is of order $O(1/\varepsilon)$ in the channels
- Accordingly, $j^\varepsilon = O(1/\varepsilon^3)$ in L^∞ and $j^\varepsilon = O(1)$ in L^1

Geometric flow rule: The density satisfies

$$j(x_1) = \frac{\alpha}{L}\big(v(x_1) - u(x_1,0)\big)$$

Mass conservation:

$$j = V\partial_1^2 v + V\omega^2\, v$$

Conclusions

- **Study: Helmholtz equation in a perforated domain**
- $O(1)$ **effect not present,** $u^\varepsilon \to u$
- $O(\varepsilon)$ **effect expressed with a limit system for** v
- **The proof uses a** $W^{1,1}(\Omega_\varepsilon)$ **bound and limit measures**

Thank you!