Homogenization of oil trapping equations

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Modelling two-phase flow

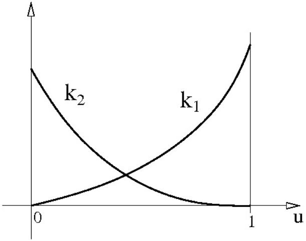
A porous material

Variables

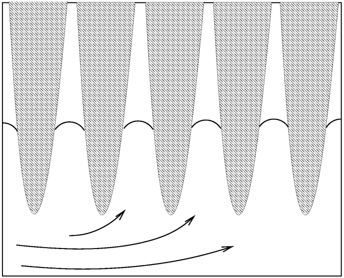
Pressure of oil	$p_1 =: p$
Pressure of water	p_2
Saturation of oil	$u_1 =: u$
Saturation of water	$u_2 = 1 - u$
Capillary pressure	$p_c(u)$
Permeabilities	$k_i = k_i(x, u)$

Equations

$$\partial_t u = \nabla \cdot (k_1(u) \nabla p_1)$$
$$-\partial_t u = \nabla \cdot (k_2(u) \nabla p_2)$$
$$p_1 - p_2 = p_c(u)$$

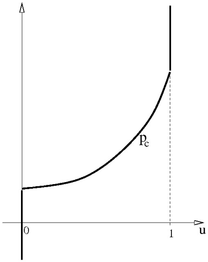


The capillary pressure



Entry of water in hydrophobic material

The capillary pressure curve is **multi-valued.** In particular: At saturation 1 *every* large pressure can be attained.



Capillary pressure law

$$p_1 - p_2 \in p_c(u),$$

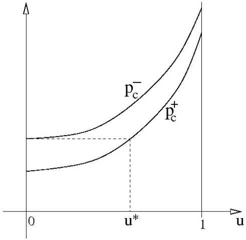
with $p_c = p_c(x, u)$, $x \in \Omega$, $u \in [0, 1]$.

Oil trapping

Interface condition.
The pressures p_1 and p_2 have no jumps. We therefore demand:

$$p_c(.,u(.)) \text{ is continuous.}$$

p_c^- law	p_c^+ law
$u \approx 0$ $p \approx p^*$ $k \approx 0$	$u \approx u^*$ $p \approx p^*$ $k > 0$



Second interface condition for

$$\partial_t u = \nabla \cdot (k_1(u) \nabla p_1)$$

is the continuity of the flux,

$$k_1(u)\partial_n p_1|_{left} = k_1(u)\partial_n p_1|_{right}$$

Both conditions were rigorously derived by *Bertsch, Dal Passo, van Duijn.*

One dimensional equations

Adding the equations yields, with $K(u) = k_1(u) + k_2(u)$,

$$\nabla \cdot (K(x,u)\nabla p - k_2(u)\nabla[p_c(u)]) = 0.$$

In the one-dimensional case

$$K(x,u)\partial_x p - k_2(u)p_c'(u)\partial_x u = -q_0$$

for some $q_0 \in \mathbb{R}$. This results in

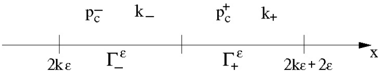
$$k_1 \partial_x p = -\frac{k_1}{K} q_0 + \frac{k_1 k_2}{K} \partial_x u$$

1D-Equations

$$\partial_t u + \partial_x F = 0$$
$$F = f(x,u) - g(x,u)\partial_x u$$

$f(u)$ and $g(u)$ are degenerate, both $\sim k_1(u) \sim u^2$.

Setting of the homogenization problem



Γ_-^ε: fine material
Γ_+^ε: coarse material
permeabilities are $k_+ > k_-$
capillary pressures $p_c^+(s) < p_c^-(s)$

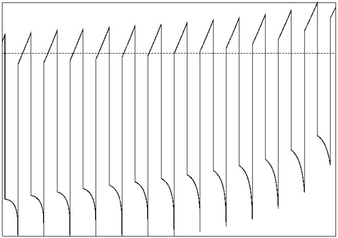
Equations

$$\partial_t u^\varepsilon + \partial_x F^\varepsilon = 0 \text{ on } \Gamma^\varepsilon = \Gamma_-^\varepsilon \cup \Gamma_+^\varepsilon$$
$$F^\varepsilon = f^\varepsilon(x, u^\varepsilon) - g^\varepsilon(x, u^\varepsilon)\partial_x u^\varepsilon$$
$$F^\varepsilon \text{ and } p_c^\varepsilon(x, u^\varepsilon) \text{ continuous in } \mathbb{Z}\varepsilon$$

with degenerate $f^\varepsilon(u)$ and $g^\varepsilon(u)$.

Effect: In Γ_+^ε the oil remains trapped.

When u^ε reaches the value u^* in Γ_+^ε, the saturation vanishes in Γ_-^ε. Because of $k_1(0) = 0$, no further flow is possible.



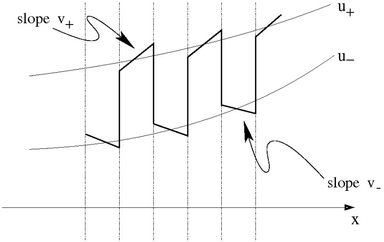
Guessing homogenized equations

u^ε with average u^0, $\partial_x u^\varepsilon$ with average v^0.

$$u_+ + u_- = 2u^0$$
$$p_c^+(u_+) = p_c^-(u_-)$$

$u_\pm$ with average slope $u_{\pm,x}$

$$u_{+,x} + u_{-,x} = 2v^0$$
$$\partial_u p_c^+(u_+)\, u_{+,x} = \partial_u p_c^-(u_-)\, u_{-,x}$$



The microscopic slopes (v_+, v_-) satisfy

$$f^+(u_+) - g^+(u_+)v_+ = f^-(u_-) - g^-(u_-)v_-$$
$$\partial_u p_c^+(u_+)v_+ + \partial_u p_c^-(u_-)v_- = \partial_u p_c^+(u_+)\, u_{+,x} + \partial_u p_c^-(u_-)\, u_{-,x}$$

Effective equations

One determines, starting from u^0 and v^0:
First $u_\pm$, then $u_{\pm,x}$, then $v_\pm$. Now we can determine the flux as

$$\mathcal{F}(u^0, v^0) := f^+(u_+) - g^+(u_+)v_+$$

Result (wishful thinking). Let u^ε be a sequence of solutions to the ε-problem with

$$u^\varepsilon \rightharpoonup u^0, \qquad F^\varepsilon \rightharpoonup F^0 \quad \text{in } L^2.$$

Then

$$\begin{aligned} &\partial_t u^0 + \partial_x F^0 = 0 \\ &\text{with } F^0 = \mathcal{F}(u^0, \partial_x u^0). \end{aligned}$$

Note: Effective equations were derived by formal asymptotics:
First equations in *J.van Duijn, A. Mikelić, and S. Pop*,
other equations in *J.van Duijn, H. Eichel, R. Helmig, and S. Pop*.

Homogenization for a positive saturation

Under the assumption $u^\varepsilon \geq \delta > 0$ on $(0,T) \times \Omega$: **ok!**

- A priori estimate $\|u^\varepsilon\|_{H^1(\Gamma^\varepsilon)} \leq c(\delta)$
- Two-scale convergence, characterize limit functions

$$u^\varepsilon \rightharpoonup u_0(x,t,y) = u_-(x,t)\mathbf{1}_{(0,1)}(y) + u_+(x,t)\mathbf{1}_{(1,2)}(y)$$
$$\partial_x u^\varepsilon \mathbf{1}_{\Gamma^\varepsilon} \rightharpoonup v_0(x,t,y) = v_-(x,t)\mathbf{1}_{(0,1)}(y) + v_+(x,t)\mathbf{1}_{(1,2)}(y)$$

- For non-linear terms use a compactness result:

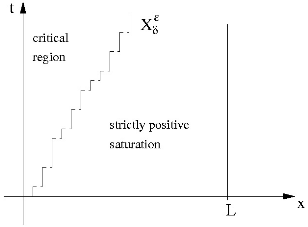
Lemma (Compactness)

For all $h \in C^0([0,1],\mathbb{R})$ holds

$$h(u^\varepsilon(x))\mathbf{1}_-^\varepsilon(x) - h(u_-(x))\mathbf{1}_-^\varepsilon(x) \to 0 \textit{ strongly in } L^2.$$

The free boundary problem

We study, for $\delta > 0$ and $\varepsilon > 0$, the free boundary separating the region of uniformly positive saturation from the rest.



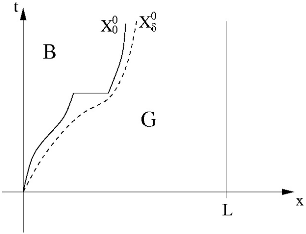
$$X_\delta^\varepsilon(t) = \inf \{x \in (0, L) \cap (2\varepsilon\mathbb{Z} + \varepsilon) : \; u^\varepsilon(x - 0, t) \geq \delta\}$$

Proposition.

1. The map $t \mapsto X_\delta^\varepsilon(t)$ is monotonically non-decreasing.
2. The following limits hold pointwise for almost every t,

$$X_\delta^0(t) = \lim_{\varepsilon \to 0} X_\delta^{\varepsilon_k}(t), \qquad X_0^0(t) = \lim_{\delta \to 0} X_{\delta_m}^0(t).$$

Limit equations with the free boundary description



Let X be the upper semicontinuous representative of X_0^0.

We decompose $(0, T) \times \Omega$ into the *good* region $G := \{(x, t) : x > X(t)\}$ and a remainder, the region B.

One shows ...

- Everywhere holds $\partial_t u^0 = \partial_x F^0$
- In G holds $F^0 = \mathcal{F}(u^0, \partial_x u^0)$
- In B holds $F^0 = 0$ and $u^0 = u^*/2$
- $\partial_x u^0$ is a non-negative measure, concentrated on $\partial B \cap \partial G$ ... and the singular part vanishes.

Theorem and Conclusions

Theorem

Let $(u^\varepsilon, F^\varepsilon)$ *be entropy solutions. Then, for a subsequence* $\varepsilon \to 0$

$$u^\varepsilon \rightharpoonup u^0 \text{ in } L^\infty_{w*}, \quad F^\varepsilon \rightharpoonup F^0 \text{ in } L^2 \text{ weakly.}$$

The limits satisfy the conservation law $\partial_t u^0 + \partial_x F^0 = 0$ *in the distributional sense. There holds* $\partial_x u^0 \in L^1(\Omega_T)$ *and*

$$F^0 = \mathcal{F}(u^0, \partial_x u^0) \text{ almost everywhere in } \Omega_T.$$

Conclusions. We have derived an effective limit equation. In particular:

1. We have determined the correct formal calculation
2. The ε-solutions may touch 0, but the effect vanishes in G for $\varepsilon \to 0$
3. The limit equation allows to decide the question $|B| > 0$.