Meta-Materials for light: Homogenization of Maxwell equations

Workshop “Variational Views in Mechanics and Materials”

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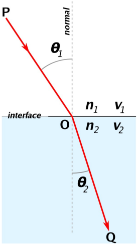
Is $\frac{7}{6}$ a small number?

Shortest Paths

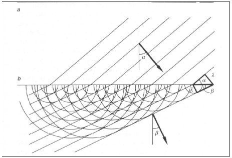
Fermat's principle of **the fastest path:**

Light finds the fastest way to reach the destination,

$$\frac{\sin \Theta_1}{\sin \Theta_2} = \frac{v_1}{v_2} = \frac{n_2}{n_1}$$



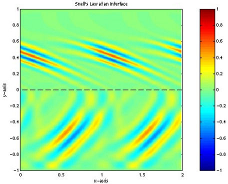
Wave equation



Huygens' principle of **superpositions**

Wave equation

$$\partial_t^2 u = \Delta u$$



Numerical solution

Maxwell's Equations

Variables:

- Electric field E
- Magnetic field H

Simplification:

- Time harmonic solutions

$$H, E \sim e^{-i\omega t}$$

Remarks:

- Vacuum: $\mu = \varepsilon = 1$
- Material parameter

$$\operatorname{Im} \varepsilon \leftrightarrow \text{ conductivity}$$

Maxwell's Equations

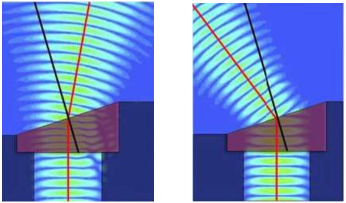
$$\begin{aligned} \text{curl } E &= \quad i\omega\mu H \\ \text{curl } H &= -i\omega\varepsilon E \end{aligned}$$

Negative index of refraction

Veselago (1968)

Properties of materials with negative index, Maxwell equations

If $n_1 > 0$ and $n_2 < 0$, then light should be refracted "backward".

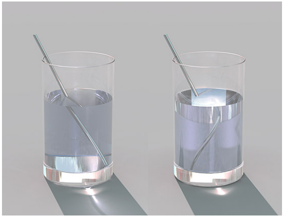


Solutions for positive and negative index

But ... in Maxwell's Equations

- $\mathrm{Re}\,\varepsilon < 0$ possible
- μ is always 1
- $\mathrm{Re}\,\mu\varepsilon < 0$: light can not travel in the medium

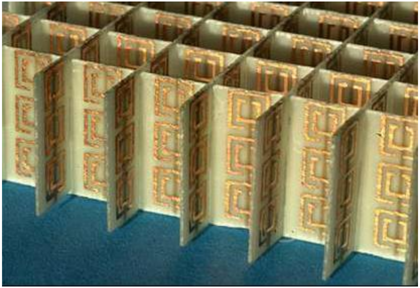
Negative Index: ε **and** μ negative!



Computer grafics: Negative refraction

Experimental construction of Meta-Materials

- Pendry et al. ($\sim$ 2000) suggest a split ring construction



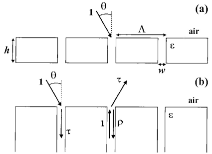
A negative index meta-material

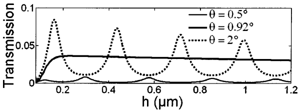
- Experiments confirm the negative index

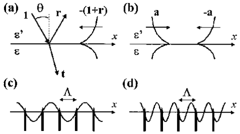
Transmission through sub-wavelength holes

Observation: Light hits a metal layer with holes. Even though the holes have sub-wavelength dimensions, the light can exit at the other side.

- Ebbesen, T. W. and Lezec, H. J. and Ghaemi, H. F. and Thio, T. and Wolff, P. A. *Extraordinary optical transmission through sub-wavelength hole arrays,* Letters to Nature 391, 1998.
- Porto, J. A. and Garcia-Vidal, F. J. and Pendry, J. B. *Transmission Resonances on Metallic Gratings with Very Narrow Slits,* Phys. Rev. Lett. 14, 1999.
- Mary, A. and Rodrigo, Sergio G. and Martin-Moreno, L. and Garcia-Vidal, F. J., *Holey metal films: From extraordinary transmission to negative-index behavior*, Physical Review B 80, 2009.
- Cao, Qing and Lalanne, Philippe *Negative Role of Surface Plasmons in the Transmission of Metallic Gratings with Very Narrow Slits*, Phys. Rev. Lett. 5, 2002.
- P. Lalanne, C. Sauvan, J. P. Hugonin, J. C. Rodier, and P. Chavel *Perturbative approach for surface plasmon effects on flat interfaces periodically corrugated by subwavelength apertures* Physical Review B 68, 2003







Surface plasmons: A mathematicians view

We consider the Helmholtz equation $\nabla \cdot (a(x)\nabla u(x)) = -k^2 u(x)$ or even

$$\nabla \cdot (a(x)\nabla u(x)) = 0$$

Let a be $+1$ for $x_1 > 0$ and -1 for $x_1 < 0$, $\omega > 0$ arbitrary

$x_1 < 0$	$x_1 > 0$
$a = -1$	$a = +1$
$u(x_1, x_2) = \exp(\omega x_1)\sin(\omega x_2)$	$u(x_1, x_2) = \exp(-\omega x_1)\sin(\omega x_2)$
Then u is	harmonic
and	continuous
and $a(x)\partial_{x_1} u(x)$ is	continuous

Similarly, solutions of the Helmholtz equation can be obtained

$\longrightarrow$ we found a wave solution that localizes at the interface

Time-harmonic Maxwell Equations

Later, $\eta > 0$ stands for the size of the holes ...

$$\begin{aligned} \text{curl } E_\eta &= \quad i\omega\mu_0 H_\eta \\ \text{curl } H_\eta &= -i\omega\varepsilon_\eta\varepsilon_0 E_\eta \end{aligned}$$

Wave number k and wavelength $\lambda = \frac{2\pi}{k}$. Further assumptions:

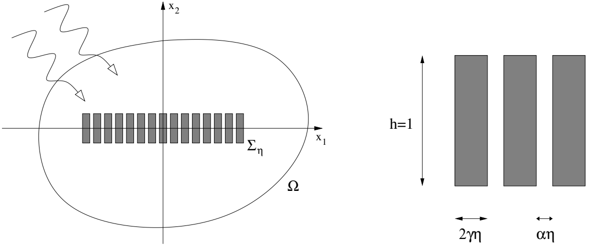
- invariance in direction x_3
- magnetic transverse polarization $H = (0, 0, u)$

2D Helmholtz equation for $H_\eta = (0, 0, u_\eta)$ with $u_\eta = u_\eta(x_1, x_2)$

$$\nabla \cdot \left(\frac{1}{\varepsilon_\eta} \nabla u_\eta \right) = -k^2 u_\eta.$$

Question: What is the behavior of u_η in the limit $\eta \to 0$?

Geometry and permittivity



The domain Ω: Maxwell equations are solved
Rectangle R: The original shape of the metal
Union of small rectangles Σ_η: The metal part after cutting holes

$$\varepsilon_\eta(x) = \begin{cases} \dfrac{\varepsilon_r}{\eta^2} & \text{for } x \in \Sigma_\eta \\ 1 & \text{for } x \notin \Sigma_\eta \end{cases}$$

Note: $|\varepsilon_\eta|$ is huge in the metal part!

First thoughts on the system

With $a_\eta = 1/\varepsilon_\eta$ (order η^2 in the metal) we must study

$$\nabla \cdot (a_\eta \nabla u_\eta) + k^2\, u_\eta = 0\,.$$

1. Outside the metal: $a_\eta \equiv 1 \longrightarrow$ no oscillations $\longrightarrow \nabla_y u = 0$
2. In the metal: $a_\eta = \eta^2 \varepsilon_r^{-1} \longrightarrow \nabla_y \cdot (\varepsilon_r^{-1} \nabla_y u) + k^2\, u = 0$

With aperture $\alpha \in (0,1)$ and metal thickness $2\gamma = 1 - \alpha$:
Define $\Psi : \mathbb{R} \to \mathbb{C}$ as the continuous, 1-periodic solution of

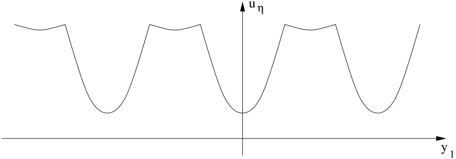
$$\begin{aligned} \partial_z^2 \Psi(z) &= -k^2 \varepsilon_r \Psi(z) && \text{for } z \in (-\gamma, \gamma) && \text{(metal)} \\ \Psi(z) &= 1 && \text{for } z \in [-1/2, 1/2] \setminus (-\gamma, \gamma) && \text{(void)} \end{aligned}$$

Ψ is given by

$$\Psi(z) = \begin{cases} \frac{\cosh(k\sigma z)}{\cosh(k\sigma\gamma)} & \text{for } |z| \le \gamma \\ 1 & \text{for } \gamma < |z| \le 1/2 \end{cases}$$

Qualitative behavior of solutions

We expect (in the rectangle R): $u_\eta \approx U(x_1, x_2)\Psi(x_1/\eta)$
In particular, in the single slit: $u_\eta \approx U(x_2)$



Here: $\varepsilon_r < 0$ real

Equation in the single slit is (for some $\tau \in \mathbb{C}$)

$$\frac{\partial^2}{\partial x_2^2} U = -k^2 \tau^2 U$$

Qualitative Argument: The second derivative $\partial_{x_1}^2 u_\eta$ in the slit is proportional to the values at the metal interfaces.

- Solutions $U(x_2)$ are $\cos(\tau k x_2)$ and $\sin(\tau k x_2)$
- For height $h > 0$ in resonance with τ: Upper and lower boundary coupled

The effective system

Original system: with $a_\eta = 1/\varepsilon_\eta$

$$\nabla \cdot (a_\eta \nabla u_\eta) \;+\; k^2\, u_\eta \;=\; 0 \quad \text{in } \Omega$$

Limit system: (loosely stated)

$$\nabla \cdot (a_{\text{eff}} \nabla U) \;+\; k^2 \mu_{\text{eff}}\, U \;=\; 0 \quad \text{in } \Omega$$

$a_{\text{eff}} : \mathbb{R}^2 \to \mathbb{R}^{2\times 2}$ and $\mu_{\text{eff}} : \mathbb{R}^2 \to \mathbb{C}$ are effective coefficients

$$a_{\text{eff}}(x) := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \mu_{\text{eff}}(x) := 1 \qquad \text{for } \; x \in \mathbb{R}^2 \setminus R$$

$$a_{\text{eff}}(x) := \begin{pmatrix} 0 & 0 \\ 0 & \alpha \end{pmatrix} \quad \text{and} \quad \mu_{\text{eff}}(x) := \beta \qquad \text{for } \; x \in R$$

$\alpha > 0$: aperture volume = relative slit width

$$\beta := \int_{-1/2}^{1/2} \Psi(z)\, dz = \frac{2}{k\sigma} \frac{\sinh(k\sigma\gamma)}{\cosh(k\sigma\gamma)} + \alpha \in \mathbb{C}\,.$$

Theorem (Bouchitté and S., 2012)

Assume $\beta \neq 0$, $\varepsilon_r \neq 0$, $a_\eta := \varepsilon_\eta^{-1}$ (small in the metal). Consider

$$\nabla \cdot j_\eta = -k^2 u_\eta \,, \qquad j_\eta = a_\eta \nabla u_\eta \,,$$

with limits $u_\eta \rightharpoonup u$ and $j_\eta \rightharpoonup j$ in $L^2(\Omega)$ for $\eta \to 0$. Set

$$U(x) := \begin{cases} u(x) & \text{for } x \in \Omega \setminus R \\ \beta^{-1} u(x) & \text{for } x \in R \end{cases}$$

There holds $\partial_{x_2} U \in L^2_{\mathrm{loc}}(\Omega)$ and

$$j = \begin{cases} (\partial_{x_1} U, \partial_{x_2} U) & \text{in } \Omega \setminus \overline{R} \\ (0,\ \alpha\, \partial_{x_2} U) & \text{in } R \end{cases}$$

and

$$\nabla \cdot j \;=\; -k^2\, u \text{ in } \Omega$$

Method of proof

1.) Two-scale convergence: $u_\eta \to u_0(x_1, x_2, y_1)$ with $(u_\eta \rightharpoonup u)$

$$u_0(x,y) = \begin{cases} u(x) & \text{for } x \notin R \\ \beta^{-1} u(x)\, \Psi(y_1) & \text{for } x \in R \end{cases}$$

2.) Two-scale convergence: $j_\eta \to j_0(x, y)$ with $(j_\eta \rightharpoonup j)$

$$j_0(x,y) = \begin{cases} j(x) & \text{for } x \notin R \\ \alpha^{-1}\, j_2(x)\, e_2\, \mathbf{1}_{\{|y_1|>\gamma\}} & \text{for } x \in R \end{cases}$$

3.) The distributional derivatives of U satisfies $\partial_{x_2} U \in L^2(\Omega)$.
Furthermore

$$j(x) = \begin{cases} (\partial_{x_1} U(x), \partial_{x_2} U(x)) & \text{for } x \notin \overline{R} \\ (0,\, \alpha\, \partial_{x_2} U(x)) & \text{for } x \in R \end{cases}$$

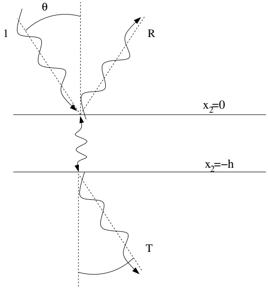
To show this relation: Consider only the void!

Transmission properties

Ansatz ($R \in \mathbb{C}$ for reflection, $T \in C$ for transmission):

$$U(x_1, x_2) = \begin{cases} e^{ik(\sin(\theta)x_1 - \cos(\theta)x_2)} + Re^{ik(\sin(\theta)x_1 + \cos(\theta)x_2)} & \text{for } x_2 > 0, \\ (A_1 \cos(\tau k x_2) + A_2 \sin(\tau k x_2))\, e^{ik(\sin(\theta)x_1} & \text{for } 0 > x_2 > -h \\ Te^{ik(\sin(\theta)x_1 - \cos(\theta)(x_2+h))} & \text{for } -h > x_2. \end{cases}$$

$\tau := \sqrt{\beta/\alpha}$ reflects the equation $\partial_{x_2}^2 U = -k^2\tau^2 U$ in the structure



At the (horizontal) interfaces:

- continuity of U
- continuity of $j \cdot e_2$

Results on transmission properties

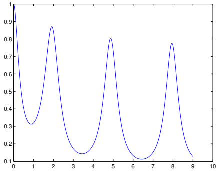
A calculation with the transfer matrix provides for $T \in \mathbb{C}$

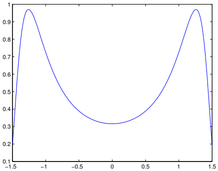
$$T = \left(\cos(\tau k h) - \frac{i}{2} \left[\frac{\alpha \tau}{\cos(\theta)} + \frac{\cos(\theta)}{\alpha \tau} \right] \sin(\tau k h) \right)^{-1}$$

Physical values taken from Qing and Lalanne in non-dimensional form ($h = 1$):

$\eta = 7/6$	$\alpha = 1/7$	$\gamma = 3/7$
$\lambda = 15/6$	$k = 2\pi/\lambda \approx 2.51$	$\varepsilon_\eta = (0.12 + 3.7i)^2$

Explicit formulas for
$\beta = \beta(\sigma, k, \alpha)$
$\tau = \tau(\beta, \alpha)$
$T = T(\tau, k)$





Left: $T = T(k)$ for normal incidence, $\theta = 0$.

Right: $T = T(\theta)$ for wave-number $k = 0.8$.

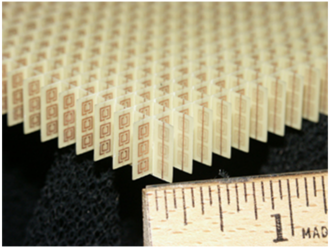
Conclusions on transmission

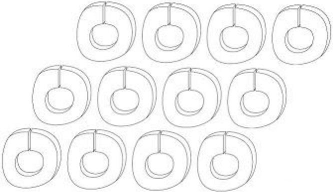
- We analyze: metal with large permittivity (η^{-2}) and small holes (η)
- Effective system is Maxwell-type, permittivity in x_1-direction is $+\infty$
- Natural field is U, not u (field outside the metal).
  The field U satisfies the continuity condition!
- Effective equations show the astonishing transmission $T \approx 1$

Negative index Meta-Materials

$(\mathrm{Re}\,\mu < 0)$

- Lamacz, A. and Schweizer, B., SIAM J. Math. Anal. 2013
- Bouchitté, G. and Schweizer, B., SIAM J. Mult. Mod. 2010





$$\Sigma_\eta := \bigcup_k \eta(k + \Sigma_Y^\eta)$$

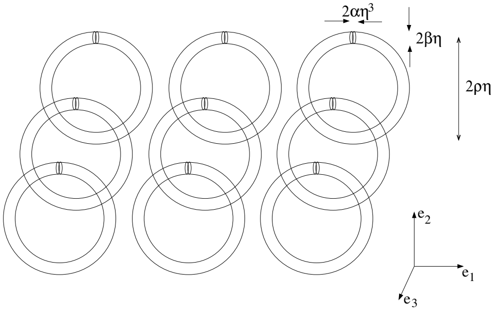
A negative index material in experiments ... and in mathematics

(H_η, E_η) solves the Maxwell system with a radiation condition at ∞.

$$\begin{aligned} \mathrm{curl}\ E_\eta &= \quad i\omega H_\eta \\ \mathrm{curl}\ H_\eta &= -i\omega\varepsilon_\eta E_\eta \end{aligned}$$

Microscopic geometry

"Many rings with thin slits"



The material parameter is

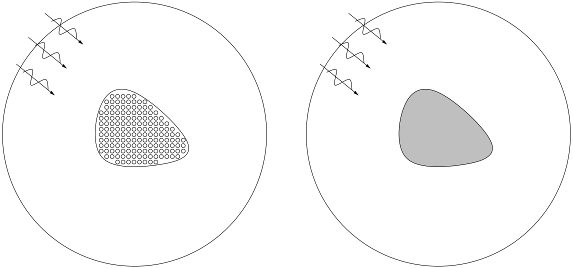
$$\varepsilon_\eta = \begin{cases} 1 + i\frac{\kappa}{\eta^2} & \text{in the rings} \\ 1 & \text{else} \end{cases}$$

The parameter η appears $3\times$:

1. thin rings / many rings
2. high conductivity
3. very thin slit

Homogenization

The aim is to replace the **complex structure** of *many* split rings with *high conductivity* by a **homogeneous Meta-material.**



The resulting equations are of the form

$$\begin{aligned} \text{curl } E &= \quad i\omega\mu_{\text{eff}} H \\ \text{curl } H &= -i\omega\varepsilon_{\text{eff}} E \end{aligned}$$

For appropriate parameters holds $\text{Re}(\mu_{\text{eff}}) < 0$.

Main result

Let ε_η be given with $\varepsilon_\eta = 1 + i\frac{\kappa}{\eta^2}$ in the rings.
Let (H_η, E_η) solve the Maxwell system with a radiation condition at ∞.

$$\operatorname{curl} E_\eta = i\omega H_\eta$$
$$\operatorname{curl} H_\eta = -i\omega\varepsilon_\eta E_\eta$$

Theorem (Bouchitté – S. 2010, Lamacz – S. 2013)

Let $(H_\eta, E_\eta) \rightharpoonup (H, E)$ in $L^2_{loc}(\mathbb{R}^3)$ for $\eta \to 0$. Then, for matrices $\hat{\mathcal{M}}_\lambda$ and $\hat{\mathcal{N}}$, the limit functions solve

$$\operatorname{curl} E = i\omega H$$
$$\operatorname{curl} (\hat{\mathcal{M}}_\lambda H) = -i\omega\hat{\mathcal{N}} E.$$

There holds $\hat{\mathcal{M}}_\lambda = \mathcal{M}_0 + \lambda(\omega,\kappa) m_0 e_3 \otimes e_3$ in Ω, with

$$\lambda(\omega,\kappa) = \frac{-\varepsilon_0\mu_0\omega^2 D_3(\omega,\kappa)}{\alpha(\pi\rho)^{-1} + \varepsilon_0\mu_0\omega^2 D_0(\omega,\kappa) - i\kappa^{-1}}.$$

Interpretation: The physical field is $\hat{H} = \hat{\mathcal{M}}_\lambda H$ such that $H = \mu^{\text{eff}}\hat{H}$

Formally, 1D-rings

$$\operatorname{curl} E_\eta = \quad i\omega H_\eta$$
$$\operatorname{curl} H_\eta = -i\omega \varepsilon_\eta E_\eta$$

Thin rings:

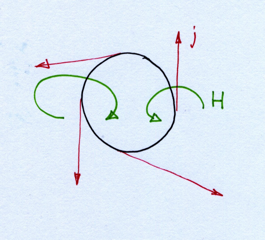
$$Y = (0,1)^3$$
$$\Sigma = S^1 \subset Y$$
$$\dim(\Sigma) = 1$$

Shape of J:

$$j_\eta := \eta \varepsilon_\eta E_\eta \rightharpoonup J$$
$$\operatorname{supp} J \subset \Sigma$$
$$\operatorname{div} J = 0$$
$$\Rightarrow J = j\tau \mathcal{H}^1 \lfloor \Sigma.$$

Shape of H:

$$\operatorname{curl} H = J$$



This determines the non-trivial part of the magnetic field.

Homogenization procedure

$$\text{curl } E_\eta = \quad i\omega H_\eta$$
$$\text{curl } H_\eta = -i\omega\varepsilon_\eta E_\eta$$

Two-scale convergence: $H_\eta(x) \to H_0(x,y)$ and $E_\eta(x) \to E_0(x,y)$ in the sense of two-scale convergence (L^2 or measures)

Loose definition: In the single periodicity cell $Y = [0,1]^3$ the solution looks like

$$H_\eta(x) \sim H_0(x,y), \qquad E_\eta(x) \sim E_0(x,y),$$

where $y \in Y$ is the local position within the cell.

The limits $H_0(x,.)$ and $E_0(x,.)$ solve the Maxwell equations, for example

$$\text{div}_y\, H_0(x,y) = 0 \text{ in } Y, \quad \text{curl}_y\, E_0(x,y) = 0 \text{ in } Y,$$
$$\text{curl}_y\, H_0(x,y) = 0 \text{ in } Y \setminus \Sigma.$$

The current J

$$\begin{aligned} \text{curl } E_\eta &= \quad i\omega H_\eta \\ \text{curl } H_\eta &= -i\omega\varepsilon_\eta E_\eta \end{aligned}$$

We additionally consider the field

$$J_\eta := \eta\varepsilon_\eta E_\eta \to J_0(x,y).$$

Then

$$\text{div} J_\eta = 0 \textbf{ implies } \text{div}_y\, J_0(x,.) = 0 \text{ in } Y$$
$$E_\eta \text{ bounded } \textbf{ implies } J_0(x,.) = 0 \text{ in } Y \setminus \Sigma$$
$$\eta \text{ curl } H_\eta = -i\omega J_\eta \textbf{ implies } \text{curl}_y\, H_0(x,.) = -i\omega J_0(x,.) \text{ in } Y.$$

Outline of the homogenization proof

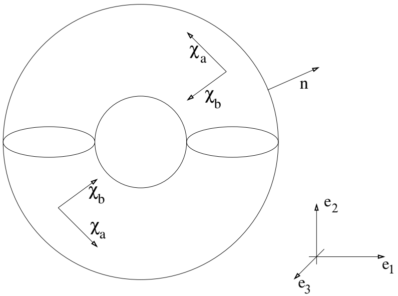
1. Introduce **current** $J_\eta := \eta \varepsilon_\eta E_\eta$ and derive estimates
2. Consider two-scale limits H_0, E_0, J_0 and **derive cell problems**
   Difficulty: Slit vanishes
3. **Analyze cell problems.**
   Difficulty: construction of the special solution H^0 "pointing through the ring"
4. Write the two-scale limit as
   $$H_0(x,y) = j(x) H^0(y) + \sum_{k=1}^{3} H_k(x) H^k(y),$$
   and **determine** j from the slit
5. Conclude the **macroscopic equation**

Result of 4: $j(x) = \lambda\, H_3(x)$ with

$$\lambda(\omega,\kappa) = \frac{-\varepsilon_0 \mu_0 \omega^2\, D_3(\omega,\kappa)}{\alpha(\pi\rho)^{-1} + \varepsilon_0\mu_0\omega^2\, D_0(\omega,\kappa) - i\kappa^{-1}}\ .$$

Step 3: 3D-cell problem without slit

We must study the 3D cell problem:



Σ the torus,
$Y = (0,1)^3$,

The H-**problem**

$$\begin{aligned} \mathrm{curl}_y\ H + i\omega\varepsilon_0 J &= 0 \text{ in } Y, \\ \mathrm{div}_y\ H &= 0 \text{ in } Y, \\ H &\text{ is periodic in } Y, \end{aligned}$$

is coupled to the J-**problem**

$$\begin{aligned} \mathrm{curl}_y\ J + \kappa\omega\mu_0 H &= 0 \text{ in } \Sigma, \\ \mathrm{div}_y\ J &= 0 \text{ in } Y, \\ J &= 0 \text{ in } Y \setminus \bar{\Sigma}. \end{aligned}$$

Lemma (Bouchitté – S., 2010)

The solution space to the above problem is four-dimensional.

Idea for the Lemma

Space for solutions:

$$X_0 := \left\{ u \in L^2_{per}(Y) : \operatorname{div} u \in L^2(Y), \operatorname{curl} u = 0 \text{ on } Y \setminus \Sigma \right\}$$

Bilinear form:

$$b_0(u, v) := \int_Y \operatorname{div} u \, \operatorname{div} \bar{v} - i k_0^2 \int_Y u \, \bar{v}$$

Regarding normalization:
On the 3D-torus $\Sigma \subset \mathbb{R}^3$ exists a vector field χ_a such that with

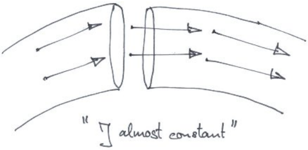
$$\operatorname{curl} \chi_a = 0, \quad \nexists \Phi : \chi_a = \nabla \Phi.$$

Normalize special solution u with

$$\int_Y u \cdot e_k = 0 \text{ for } k = 1, 2, 3, \text{ and } \int_\Sigma u \cdot \chi_a = 1.$$

Step 4: Slit analysis

The flux J_η is almost constant — across the slit!
(despite $\varepsilon_\eta = 1 + i\frac{\kappa}{\eta^2}$ in the ring)



$$E_\eta \sim \begin{cases} \frac{1}{i\omega}\frac{1}{\eta} j_0 \tau & \text{in the slit} \\ \frac{-1}{\kappa\omega}\eta j_0 \tau & \text{inside the material} \end{cases}$$

$$\int_{\text{closed ring}} E_\eta \cdot \tau = \int_{\text{disc}} e_3 \cdot \operatorname{curl} E_\eta = i\omega \int_{\text{disc}} e_3 \cdot H_\eta$$

... and in the limit $\eta \to 0$:

$$-\frac{2i}{\omega}\left(\alpha - i\frac{\rho\pi}{\kappa}\right) j_0(x) = i\omega\left(D_3 H_3(x) - D_0 j_0(x)\right)$$

This provides $j_0(x) = \lambda(\omega) H_3(x)$.

Step 5: Macroscopic equation

Use test-functions Φ from $\Phi(x,y) = \psi(x)\Theta(y)$ with $\operatorname{curl}_y \Theta = 0$ in Y, $\Theta \equiv 0$ on $\operatorname{conv}\Sigma$. Then, for $\eta \to 0$,

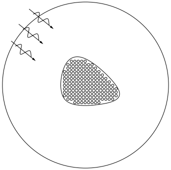
$$\int_{\mathbb{R}^3} \operatorname{curl} H_\eta \cdot \Phi = \int_{\mathbb{R}^3} (-i\omega\varepsilon_\eta E_\eta) \cdot \Phi \to \langle -i\omega E(x), \psi(x)\rangle$$

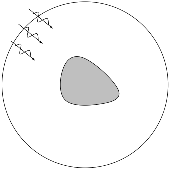
... and the left hand side equals

$$\begin{aligned} &\int_{\mathbb{R}^3} H_\eta \cdot \operatorname{curl}\Phi = \int_{\mathbb{R}^3} H_\eta(x)\Theta(x/\eta) \wedge \nabla\psi(x)\, dx \\ &\to \int_{\mathbb{R}^3}\int_Y \left\{ H(x) + \lambda(\omega) H_3(x) H^0(y) \right\} \Theta(y) \wedge \nabla\psi(x)\, dy\, dx \\ &= \int_{\mathbb{R}^3} [\hat{M} H(x)] \wedge \nabla\psi(x)\, dx = \left\langle \operatorname{curl}[\hat{M} H], \psi(x) \right\rangle \end{aligned}$$

This provides $\operatorname{curl}[\hat{M} H] = -i\omega E(x)$.

Conclusions on negative index materials





$$\begin{aligned} \text{curl } E &= \quad i\omega\mu_{\text{eff}} H \\ \text{curl } H &= -i\omega\varepsilon_{\text{eff}} E \end{aligned}$$

- split ring geometry with highly conducting rings
- 3D-scattering problem, Maxwell equations
- formulas for μ_{eff} and ε_{eff} in terms of conductivity and geometric quantities
- mathematical proofs for the homogenization result
- Calculations show: μ_{eff} can be negative (despite $\mu_0 \equiv 1$)

Thank you!

All formulas (perfect transmission):

Permittivity with $\sigma^2 = -\varepsilon_r$

$$\varepsilon_\eta(x) = \begin{cases} \dfrac{\varepsilon_r}{\eta^2} & \text{for } x \in \Sigma_\eta \\ 1 & \text{for } x \notin \Sigma_\eta \end{cases}$$

Use $a_\eta = 1/\varepsilon_\eta$

$$\beta = \frac{2}{k\sigma}\frac{\sinh(k\sigma\gamma)}{\cosh(k\sigma\gamma)} + \alpha$$

$$\tau = \sqrt{\beta/\alpha}$$

$$T = \left(\cos(\tau kh) - \frac{i}{2}\left[\frac{\alpha\tau}{\cos(\theta)} + \frac{\cos(\theta)}{\alpha\tau}\right]\sin(\tau kh)\right)^{-1}$$