Effective Helmholtz equation for domains with a perforation along an interface

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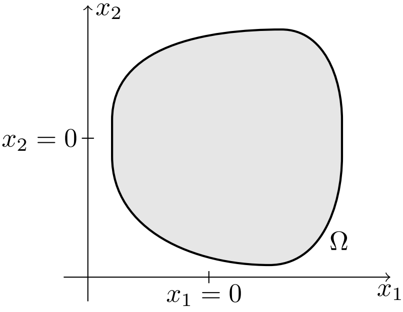
Helmholtz equation

Sound is described by the wave equation $\partial_t^2 p = \Delta p$.
The time-harmonic ansatz $p = p(x)e^{i\omega t}$ leads to the

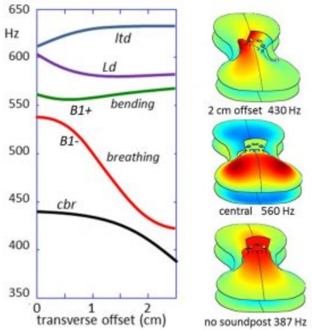
Helmholtz equation

$$-\Delta p = \omega^2 p + f \qquad \text{in } \Omega$$

Here: $f \in L^2(\Omega)$ a prescribed source



A domain $\Omega \subset \mathbb{R}^d$ for $d = 2$

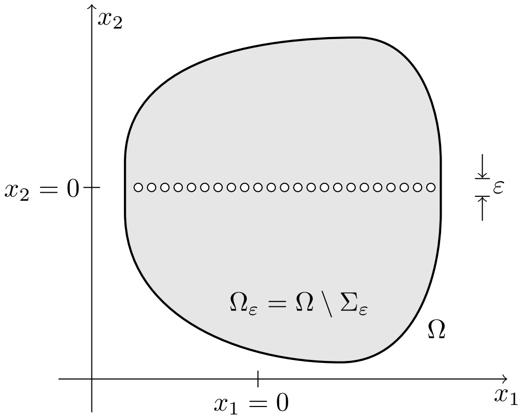


Graphic taken from: Acoustics Today

The Neumann sieve geometry

Helmholtz equation

$$-\Delta p^\varepsilon = \omega^2 p^\varepsilon + f \qquad \text{in } \Omega_\varepsilon \tag{1}$$



Dirichlet condition on $\partial\Omega$

Always: Homogeneous Neumann boundary condition on $\partial\Omega_\varepsilon \setminus \partial\Omega$

What is the effect of a perforation along a plane?

Notation

Inclusions: Index $k \in \mathbb{Z}^{d-1}$. The single inclusion is

$$\Sigma_k^\varepsilon := \varepsilon\left(\Sigma + (k,0)\right) \quad \text{for} \quad k \in \mathbb{Z}^{d-1}$$

Number of inclusions $\sim \varepsilon^{-(d-1)}$

Perforated domain:

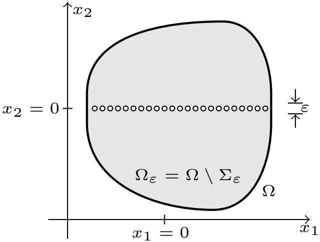
$$\Sigma_\varepsilon := \bigcup_{k \in I_\varepsilon} \Sigma_k^\varepsilon \qquad \Omega_\varepsilon := \Omega \setminus \bar{\Sigma}_\varepsilon$$

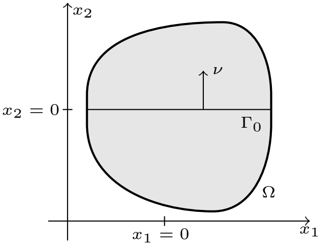
Limit geometry: The perforation Σ_ε is located along the submanifold

$$\Gamma_0 := \left(\mathbb{R}^{d-1} \times \{0\}\right) \cap \Omega$$

Normal vectors: $n = n_\varepsilon(x)$ the outer normal of Ω_ε

The interface has the upward pointing normal $\nu = e_d$





A surprising observation

Helmholtz equation $-\Delta p^\varepsilon = \omega^2 p^\varepsilon + f$, assume $\|p^\varepsilon\|_{L^2} \le C$

Extension: $\mathcal{P}_\varepsilon : L^2(\Omega_\varepsilon) \to L^2(\Omega)$ maps a function to its trivial extension

- Multiply equation with p^ε, Poincaré $\longrightarrow$ $\|p^\varepsilon\|_{H^1(\Omega_\varepsilon)} \le C$
- $\mathcal{P}_\varepsilon p^\varepsilon \rightharpoonup p$ and $\mathcal{P}_\varepsilon(\nabla p^\varepsilon) \rightharpoonup g$ in $L^2(\Omega)$, $g = \nabla p \in \Omega \setminus \Gamma_0$
- Poincaré to compare p across layer $\longrightarrow$ $[p] = 0$, $g = \nabla p$ in Ω
- Limit in equation: $\int_\Omega \nabla p \cdot \nabla \varphi = \omega^2 \int_\Omega p\,\varphi + \int_\Omega f\,\varphi$

Result:

The limit function p is the $H^1(\Omega)$-solution of

$$-\Delta p = \omega^2 p + f \qquad \text{in } \Omega \tag{2}$$

$\longrightarrow$ **The perforation has no effect!** (at order 1)

For a priori bound in L^2 we assume:
ω^2 is not a Dirichlet eigenvalue of $-\Delta$ on Ω:

$$\omega^2 \notin \sigma(-\Delta)$$

The leading order limit

Theorem (Trivial limit and rate of convergence, DHS 2017)

Let p^ε *be solutions to* (1) *and let the dimension be* $d = 3$
With the unique weak solution $p \in H_0^1(\Omega)$ *of* (2) *holds*

$$\mathcal{P}_\varepsilon p^\varepsilon \to p \quad \text{and} \quad \mathcal{P}_\varepsilon \nabla p^\varepsilon \rightharpoonup \nabla p \quad \text{in } L^2(\Omega)$$

Let f *have the regularity* $H^1 \cap C^\alpha$, $\alpha > 0$, *and let* $\partial\Omega$ *be of class* C^3
For a constant $C = C(f)$ *holds*

$$\|p - \mathcal{P}_\varepsilon p^\varepsilon\|_{L^2(\Omega)} + \|\nabla p - \mathcal{P}_\varepsilon \nabla p^\varepsilon\|_{L^2(\Omega)} \leq C\varepsilon^{1/2} \tag{3}$$

C. Dörlemann, M. Heida, and B. Schweizer. Transmission conditions for the Helmholtz-equation in perforated domains. *Vietnam J. Math.*, 45(1-2):241–253, 2017

The first order limit

p^ε: solution of (1) on Ω_ε p: solution of (2) on Ω

Define the corrector

$$v^\varepsilon := \frac{p^\varepsilon - p}{\varepsilon} \tag{4}$$

Assume $v^\varepsilon \to v$. What are the equations for v?

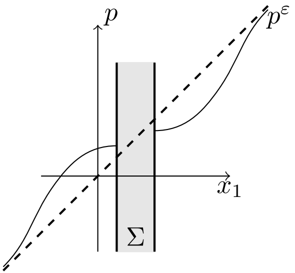
Orders of magnitude

- ∇p is smooth, order $O(1)$ around inclusion
- $n \cdot \nabla v^\varepsilon = -\frac{1}{\varepsilon} n \cdot \nabla p$ of order $O(\varepsilon^{-1})$
- v^ε has variations $O(1)$

Functions spaces

bad: $\|\nabla v^\varepsilon\|_{L^2(\Omega_\varepsilon)} \to \infty$ expected

good: $\|\nabla v^\varepsilon\|_{L^1(\Omega_\varepsilon)} \leq C$ possible



p and p^ε near an obstacle

Let's follow a classical advice ...

divide et impera!

Assumption

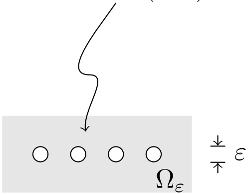
For some $C > 0$, independent of ε:

$$\|v^\varepsilon\|_{W^{1,1}(\Omega_\varepsilon)} \le C \tag{5}$$

Two questions

1. What are the equations for v?
2. Why should v^ε satisfy (5)?

$$v^\varepsilon = O(1)$$
$$\nabla v^\varepsilon = O(\varepsilon^{-1})$$



Orders of magnitude near an obstacle

Assumption (5) implies for $q > 1$ and $v \in L^1(\Omega)$:

- $\mathcal{P}_\varepsilon v^\varepsilon \overset{*}{\rightharpoonup} v\, d\mathcal{L}^d$ weak-$*$ as measures
- $\mathcal{P}_\varepsilon \nabla v^\varepsilon \overset{*}{\rightharpoonup} \nabla v + \mu$ for some measure μ with $\operatorname{supp}(\mu) \subset \Gamma_0$
- $v \in L^q_{\text{loc}}(\Omega)$ and $\mathcal{P}_\varepsilon v^\varepsilon \to v$ in $L^1_{\text{loc}}(\Omega)$
- $v \in W^{1,1}(\Omega \setminus \Gamma_0)$

The main result

Theorem (Effective system for the corrector, S. 2018)

p^ε and p as above (solutions to Helmholtz), corrector v^ε given by

$$v^\varepsilon = \frac{p^\varepsilon - p}{\varepsilon}$$

Assume the $\varepsilon^{1/2}$-L^2-bound (3), *the $W^{1,1}$-bound* (5), *and $\mathcal{P}_\varepsilon v^\varepsilon \to v$*
Then $v \in W^{1,1}(\Omega \setminus \Gamma_0)$ is the unique solution of

$$\begin{aligned} -\Delta v &= \omega^2 v && \text{in } \Omega \setminus \Gamma_0 \\ [v] &= J \cdot \nabla p && \text{on } \Gamma_0 \\ [\partial_\nu v] &= \nabla \cdot (G \nabla p) && \text{on } \Gamma_0 \end{aligned} \tag{6}$$

The matrices $G \in R^{d\times d}$ and $J \in \mathbb{R}^d$ are given by cell problems

Result: weak coupling! One solves first system for p. The corrector is given by a Helmholtz equation that involves $p|_{\Gamma_0}$ and $\nabla p|_{\Gamma_0}$ as data

B. Delourme, H. Haddar, and P. Joly. Approximate models for wave propagation across thin periodic interfaces. *J. Math. Pures Appl. (9)*, 98(1):28–71, 2012.

Cell problems

$$Y := \left(-\frac{1}{2}, \frac{1}{2}\right)^{d-1}_{\text{per}} \times \mathbb{R} \qquad\qquad Z := Y \setminus \Sigma$$

The Lipschitz domain Σ (obstacle) is compactly contained

Definition: Cell problem

Given $\xi \in \mathbb{R}^d$, seek $w \in H^1_{\text{loc}}(Z)$ such that

$$\begin{aligned} -\Delta w &= 0 && \text{in } Z \\ \partial_n w &= n \cdot \xi && \text{on } \partial\Sigma \end{aligned} \tag{7}$$

$n : \partial\Sigma \to \mathbb{R}^d$ is the exterior normal of Z

Lemma: Existence and uniqueness for cell problem

For $\xi \in \mathbb{R}^d$ there exists a (unique up to constants) solution w,

$$w \in \dot{H}(Z) := \left\{ w \in H^1_{\text{loc}}(Z) \,\middle|\, \nabla w \in L^2(Z) \right\}$$

$$\|w\|^2_{\dot{H}} := \int_{Z \cap \{|y_d| < 1\}} |w|^2 + \int_Z |\nabla w|^2$$

Effective coefficients

Recall:
$\partial_n w = n \cdot \xi$ on $\partial\Sigma$
$[v] = J \cdot \nabla p$
$[\partial_\nu v] = \nabla \cdot (G \nabla p)$

For arbitrary $\xi \in \mathbb{R}^d$ and $w = w_\xi$

"Gradient": $G \in \mathbb{R}^{d\times d}$

$$G\,\xi := \int_Z \nabla w \in \mathbb{R}^d$$

"Jump": $J \in \mathbb{R}^d$

$$J \cdot \xi := -\lim_{\zeta\to\infty} \int_{\{y_d=\zeta\}} w + \lim_{\zeta\to-\infty} \int_{\{y_d=\zeta\}} w \in \mathbb{R}$$

Lemma (Structural properties)

The matrix G and the vector J are well defined. They have the form

$$G = \begin{pmatrix} G_\tau & J_\tau \\ 0 & -|\Sigma| \end{pmatrix} \qquad J = \begin{pmatrix} J_\tau \\ \gamma \end{pmatrix}$$

with $G_\tau \in \mathbb{R}^{(d-1)\times(d-1)}$ symmetric and positive definite, $J_\tau \in \mathbb{R}^{d-1}$, $\gamma \in \mathbb{R}$ with $\gamma > |\Sigma|$.

Idea of the proof: Elementary unfolding

Let $\varphi \in C_c^\infty(\Omega)$ be arbitrary. Consider $V_\varphi^\varepsilon : Z \to \mathbb{R}$,

$$V_\varphi^\varepsilon(y) := \frac{1}{|I_\varepsilon|} \sum_{k \in I_\varepsilon} v^\varepsilon(\varepsilon(k+y))\, \varphi(\varepsilon(k+y))$$

Derive estimates for V_φ^ε using $\|v^\varepsilon\|_{L^2(\Omega_\varepsilon)} + \|\nabla v^\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq C\, \varepsilon^{-1/2}$:

$$\int_Z |\nabla V_\varphi^\varepsilon|^2 \leq C \int_Z \varepsilon^{d-1} \sum_k |\varepsilon^2 \nabla v^\varepsilon(\varepsilon(k+y))|^2\, dy$$

$$\leq C \int_{\Omega_\varepsilon} \varepsilon^{-d} \varepsilon^{d-1} \varepsilon^2 |\nabla v^\varepsilon(x))|^2\, dx \leq C$$

Conclude

$$V_{\varphi,0}^\varepsilon \rightharpoonup w \text{ in } \dot{H}^1(Z)$$

as $\varepsilon \to 0$. Here w is the cell-problem solution for

$$\xi := -\frac{1}{|\Gamma_0|} \int_{\Gamma_0} \nabla p\, \varphi \in \mathbb{R}^d$$

Furthermore, there holds

$$e_j \cdot \int_{\partial\Sigma_\varepsilon} n\, v^\varepsilon\, \varphi \to |\Gamma_0| e_j \cdot \int_{\partial\Sigma} n\, w$$

Main proposition

Proposition (Equations for weak limits)

p^ε, p, and v^ε as above, v and μ the limits:

$$\mathcal{P}_\varepsilon v^\varepsilon \overset{*}{\rightharpoonup} v\, d\mathcal{L}^d \quad \text{and} \quad \mathcal{P}_\varepsilon \nabla v^\varepsilon \overset{*}{\rightharpoonup} \nabla v + \mu$$

Then μ is given by

$$\mu = -G\nabla p\, \mathcal{H}^{d-1}\lfloor_{\Gamma_0} \tag{8}$$

and v satisfies the system (6).

On the proof I. An integration by parts for $j < d$:

$$\int_{\Omega_\varepsilon} \partial_j v^\varepsilon \varphi + \int_{\Omega_\varepsilon} v^\varepsilon \partial_j \varphi = e_j \cdot \int_{\partial\Sigma_\varepsilon} n\, v^\varepsilon \varphi$$

In the limit $\varepsilon \to 0$:

$$\int_\Omega \partial_j v\, \varphi + \int_\Omega e_j\, \varphi \cdot d\mu + \int_\Omega v\, \partial_j \varphi = -\int_{\Gamma_0} e_j \cdot G\nabla p\, \varphi$$

This shows

$$e_j \cdot \mu = -e_j \cdot G\nabla p\, \mathcal{H}^{d-1}\lfloor_{\Gamma_0} \tag{9}$$

On the proof II

On the proof II. Limits in the weak form of the equation

$$\int_{\Omega\setminus\Gamma_0} \nabla v \cdot \nabla\varphi + \int_\Omega \nabla\varphi \cdot d\mu \longleftarrow \int_{\Omega_\varepsilon} \nabla v^\varepsilon \cdot \nabla\varphi$$

$$= -\int_{\partial\Omega_\varepsilon} \frac{1}{\varepsilon} n \cdot \nabla p\,\varphi + \int_{\Omega_\varepsilon} \omega^2 v^\varepsilon\,\varphi$$

$$\to |\Sigma| \int_{\Gamma_0} \left(\partial_\nu^2 p\,\varphi + \partial_\nu p\,\partial_\nu\varphi\right) + \int_\Omega \omega^2 v\,\varphi$$

$\varphi \in C_c^\infty(\Omega)$ that vanish on Γ_0 and have $\partial_\nu\varphi$ arbitrary on Γ_0:

$$e_d \cdot \mu = |\Sigma|\,\partial_\nu p\,\mathcal{H}^{d-1}\lfloor_{\Gamma_0}$$

General $\varphi \in C_c^\infty(\Omega)$ yields the jump condition

$$[\partial_\nu v] = \nabla \cdot G\nabla p$$

The jump condition for values follows similarly

Proof of the $W^{1,1}$-bound

Can a function u^ε with $\partial_n u^\varepsilon = O(\varepsilon^{-1})$ be bounded in $W^{1,1}$?

Proposition (Construction of $W^{1,1}$-bounded sequences)

$R := (-1,1)^{d-1} \times (-h,h)$ *a cuboid,*
$g \in C^2(\bar{R}) \cap H^3(R)$ *prescribes boundary data*
$\Sigma \subset Y$ *satisfies a regularity property (solutions in* L^∞*)*
$R_\varepsilon := R \setminus \Sigma_\varepsilon$ *the perforated domains*

Then there exists a sequence $u_\varepsilon : R_\varepsilon \to \mathbb{R}$ *of class* $H^2(R_\varepsilon)$ *such that*

$$u_\varepsilon \in L^2(R_\varepsilon) \cap W^{1,1}(R_\varepsilon)$$

$$\sigma_\varepsilon := \left(\partial_n u_\varepsilon - \frac{1}{\varepsilon} g \cdot n\right)\Big|_{\partial\Sigma_\varepsilon} \in L^\infty(\partial\Sigma_\varepsilon)$$

$$\rho_\varepsilon := \Delta u_\varepsilon \in L^\infty(R_\varepsilon)$$

are bounded in the indicated function spaces

Idea of proof: Write u_ε explicitly with second order cell solutions ψ,

$$u_\varepsilon(x) := w_j(x/\varepsilon) g_j(x) + \varepsilon \psi_{i,j}(x/\varepsilon) \partial_i g_j(x)$$

Conclusions

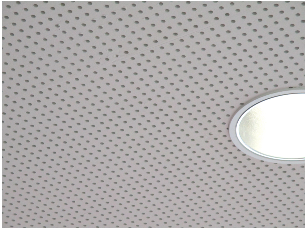
Theorem: An example where the a priori bounds are satisfied

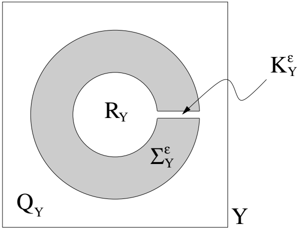
Let $\Omega = (0,1)^{d-1} \times (-h,h)$ be a cuboid, consider homogeneous Dirichlet boundary conditions on $\partial\Omega$ and let $\Sigma \subset \left(-\frac{1}{2}, \frac{1}{2}\right)^{d-1} \times \mathbb{R}$ possess reflection symmetry in every direction e_j, $j = 1, ..., d-1$.
Then the corrector v^ε satisfies the $W^{1,1}$-bound (5)

- **Helmholtz equation in a perforated domain**
- $O(1)$ **effect not present,** $p^\varepsilon \to p$
- $O(\varepsilon)$ **effect expressed with a limit system for** v
- **The proof uses a** $W^{1,1}(\Omega_\varepsilon)$ **bound and limit measures**

Outlook: Many Helmholtz resonators

Ω_ε is perforated with period $\varepsilon > 0$ ... and the single inclusion has two scales!





A.Lamacz & B.S., 2016, resonators fill an open domain

$u^\varepsilon \rightharpoonup v$ outside resonators, v solves the *effective Helmholtz equation*

$$-\nabla \cdot (A_* \nabla v) = \omega^2 \Lambda\, v \text{ in } \Omega$$

The effective coefficient is $\Lambda = Q - \frac{A}{L}\left(\omega^2 - \frac{A}{LV}\right)^{-1}$ **Any value!**

Thank you!