

# Gradient flows: Absolut essentials and manifolds

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Material for a seminar talk

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**Abstract:** This material is hopefully useful for an understanding of (the first steps of) the “geometry” work of F. Otto from 2001, [1]. We give the easiest nontrivial example for a gradient flow on a manifold.

## 1. GRADIENT FLOW IN A HILBERT SPACE

When  $X$  is a Hilbert space and  $E : X \rightarrow \mathbb{R}$  is differentiable, then the differential in a point  $u \in X$  is a map  $DE(u) : X \rightarrow \mathbb{R}$ . It is defined as

$$(1.1) \quad DE(u)\langle v \rangle := \left. \frac{d}{dt} \right|_{t=0} E(u + tv)$$

for every  $v \in X$ . With the scalar product on  $X$ , we define the gradient  $\nabla E(u) \in X$  by demanding

$$(1.2) \quad DE(u)\langle v \rangle \stackrel{!}{=} \langle \nabla E(u), v \rangle_X \quad \forall v \in X.$$

Let us investigate a solution  $u : [0, T] \ni t \mapsto u(t) \in X$ , of the gradient flow equation

$$(1.3) \quad \partial_t u = -\nabla E(u),$$

which is meant to be satisfied for every  $t \in (0, T)$ . This equation means that we are “walking in the direction of the steepest decent”. We calculate for the evolution of the energy, omitting the argument  $t$ ,

$$(1.4) \quad \frac{d}{dt}[E(u)] = DE(u)\langle \partial_t u \rangle = \langle \nabla E(u), \partial_t u \rangle_X < 0.$$

Moreover, the decay of energy is quantified. The right hand side can be written in any of these forms:

$$(1.5) \quad -\|\nabla E(u)\|^2 = -\|\partial_t u\|^2 = -\|\partial_t u\| \|\nabla E(u)\| = -\frac{1}{2}\|\partial_t u\|^2 - \frac{1}{2}\|\nabla E(u)\|^2.$$

The subsequent three examples are formal in the sense that (in two of the examples) the energy is not differentiable on the whole space  $X$ . Regarding  $\Omega \subset \mathbb{R}^n$ , we always think of a bounded Lipschitz domain.

**1.1. The  $L^2$ -gradient flow of the  $L^2$ -energy.** We consider  $X = L^2(\Omega)$  and the energy  $E : X \rightarrow \mathbb{R}$ , defined by  $E(u) = \frac{1}{2} \int_{\Omega} |u|^2$ . Then

$$(1.6) \quad DE(u)\langle v \rangle = \int_{\Omega} u \cdot v, \quad \nabla E(u) = u.$$

The corresponding gradient flow is therefore

$$(1.7) \quad \partial_t u = -u.$$

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**1.2. The  $L^2$ -gradient flow of the  $H^1$ -energy.** We consider  $X = L^2(\Omega)$  and the energy  $E : X \rightarrow \mathbb{R}$ , defined by  $E(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2$ . Then

$$(1.8) \quad DE(u)\langle v \rangle = \int_{\Omega} \nabla u \cdot \nabla v, \quad \nabla E(u) = -\Delta u.$$

The corresponding gradient flow is therefore

$$(1.9) \quad \partial_t u = \Delta u.$$

**1.3. The  $H^{-1}$ -gradient flow of the  $L^2$ -energy.** We consider  $X = H^{-1}(\Omega)$  and the energy  $E : X \rightarrow \mathbb{R}$ , defined by  $E(u) = \frac{1}{2} \int_{\Omega} |u|^2$ .

In order to prepare the analysis, we have to choose a scalar product on  $H^{-1}(\Omega) = (H_0^1(\Omega))'$ . There is a natural choice when we use the scalar product  $\langle u, v \rangle_{H^1} := \int_{\Omega} \nabla u \cdot \nabla v$  on  $H_0^1(\Omega)$ . Every element  $\lambda \in X = H^{-1}(\Omega)$  can be represented by  $U_{\lambda} \in H_0^1(\Omega)$  via  $\lambda(\varphi) = \langle U_{\lambda}, \varphi \rangle_{H^1}$  for all  $\varphi \in H_0^1(\Omega)$ . By the choice of the scalar product in  $H_0^1(\Omega)$ , the objects  $\lambda$  and  $U_{\lambda}$  are related by equation  $-\Delta U_{\lambda} = \lambda$  in the sense of distributions. The natural scalar product on  $X = H^{-1}(\Omega)$  is therefore, for  $\lambda, \mu \in X$ , given by

$$(1.10) \quad \langle \lambda, \mu \rangle_X := \langle U_{\lambda}, U_{\mu} \rangle_{H^1} = \lambda(U_{\mu}) = \int_{\Omega} \nabla U_{\lambda} \cdot \nabla U_{\mu} = \int_{\Omega} U_{\lambda} \mu,$$

where the last expression can only be written in this form when  $\mu$  is an  $L^2(\Omega)$ -function.

The differential of  $E$  is (formally, when  $u$  and  $v$  are  $L^2(\Omega)$ ):

$$(1.11) \quad DE(u)\langle v \rangle = \int_{\Omega} u \cdot v.$$

We are now in the position to calculate the gradient  $g = \nabla E(u)$ . For arbitrary  $v \in X$  (in the calculation we actually assume  $v \in L^2(\Omega)$ ):

$$(1.12) \quad \int_{\Omega} u \cdot v = DE(u)\langle v \rangle \stackrel{!}{=} \langle \nabla E(u), v \rangle_X = \langle g, v \rangle_X = \int_{\Omega} (-\Delta)^{-1}(g) v.$$

Since  $v$  was arbitrary, we find  $u = (-\Delta)^{-1}(g)$  or, equivalently,  $g = -\Delta u$ .

The gradient flow corresponding to  $X$  and  $E$  is therefore

$$(1.13) \quad \partial_t u = \Delta u.$$

Even though we have chosen another energy and another underlying space, we have the same equation as in (1.9).

## 2. ELEMENTARY DIFFERENTIAL GEOMETRY

The aim of this section is to support readers that want to understand [1]. Because of this aim, all the notation is as in [1]. There is a minimal exception: We write  $DE|_{\rho}\langle s \rangle$  instead of  $\text{diff}E|_{\rho}.s$ .

**2.1. The simplest nontrivial gradient flow on a manifold.** Guiding question: Given a manifold  $\mathcal{M}$  (the elements are denoted by  $\rho$ ) and given a function  $E : \mathcal{M} \rightarrow \mathbb{R}$ , we are interested in the gradient flow equation

$$(2.1) \quad \partial_t \rho = -\nabla E(\rho).$$

More precisely: We seek a map  $\rho : [0, T] \rightarrow \mathcal{M}$  such that (2.1) holds for almost every  $t$ .

We must ask: What exactly is meant with (2.1)? In particular: What kind of object is the gradient? We recall here some differential geometry in order to clarify these questions. With the example of  $\mathcal{M} = S^1 \subset \mathbb{R}^2$ , we illustrate the concepts.

Abstract manifold setting	Our example
Hilbert space $H$	$H = \mathbb{R}^2$
manifold $\mathcal{M} \subset H$	Our choice: $\mathcal{M} := S^1$
elements $\rho \in \mathcal{M}$	$S^1 = \{\rho \in \mathbb{R}^2 \mid \rho_1^2 + \rho_2^2 = 1\}$
tangent space $T_\rho \mathcal{M}$	$T_\rho S^1 = \{s \in \mathbb{R}^2 \mid s \cdot \rho = 0\}$
equivalent curves $\gamma$ with $\gamma(0) = \rho$	subset of vectors $\gamma'(0) \in H$
metric tensor $g = g_\rho(\cdot, \cdot)$	$g_\rho(s_1, s_2) = \langle s_1, s_2 \rangle_H = s_1 \cdot s_2$
Riemannian manifold: part of the definition of $\mathcal{M}$	submanifold of a Hilbert space: metric induced by $H$
energy functional $E : \mathcal{M} \rightarrow \mathbb{R}$	Our choice: $E(\rho) := \rho_2$ (height)
differential $DE _\rho : T_\rho \mathcal{M} \rightarrow \mathbb{R}$	$DE _\rho \langle s \rangle = \frac{d}{dt}(E \circ \gamma)(t_0)$ $\gamma(t_0) = \rho, \gamma'(t_0) = s$

TABLE 1. Embedded manifold and our example

We consider the example that is outlined in the right part of Table 1. Let us calculate the differential of  $E$  in the point  $\rho = (\cos(t_0), \sin(t_0))$ . An arbitrary tangential vector in  $\rho$  is of the form  $\mu(-\sin(t_0), \cos(t_0))$ . It is sufficient to evaluate  $DE|_\rho \langle s \rangle$  for  $s = (-\sin(t_0), \cos(t_0))$ . As a curve through  $\rho$  with derivative  $s$  we choose  $\gamma(t) := (\cos(t), \sin(t))$ . We find

$$DE|_\rho \langle s \rangle = \frac{d}{dt}(E \circ \gamma)(t)|_{t=t_0} = \frac{d}{dt} \sin(t)|_{t=t_0} = \cos(t_0).$$

The action on an arbitrary tangential vector is therefore, for  $\rho = (\cos(t_0), \sin(t_0))$ ,

$$(2.2) \quad DE|_\rho \langle \mu(-\sin(t_0), \cos(t_0)) \rangle = \mu \cos(t_0).$$

With this calculation,  $DE$  is determined.

**Definition 2.1** (Gradient). *The vector  $\nabla E(\rho)$  is the element of  $T_\rho \mathcal{M}$  such that*

$$(2.3) \quad g_\rho(\nabla E(\rho), s) = DE|_\rho \langle s \rangle$$

*holds for every  $s \in T_\rho \mathcal{M}$ .*

We continue our example, we now consider a point  $\rho = (\cos(t), \sin(t))$ . The gradient  $\nabla E(\rho)$  is a tangent vector, therefore, for some  $\lambda \in \mathbb{R}$ , there must hold  $\nabla E(\rho) = \lambda(-\sin(t), \cos(t))$ . Test vectors are also tangential vectors, we write them as  $s = \mu(-\sin(t), \cos(t)) \in T_\rho \mathcal{M}$ . The gradient is defined by

$$(2.4) \quad \lambda \mu = g_\rho(\nabla E(\rho), s) \stackrel{!}{=} DE|_\rho \langle s \rangle = \mu \cos(t).$$

Since this should hold for every  $\mu \in \mathbb{R}$ , we find  $\lambda = \cos(t)$ . We have thus determined the gradient  $\nabla E(\rho)$ :

$$(2.5) \quad \nabla E((\cos(t), \sin(t))) = \cos(t) (-\sin(t), \cos(t)).$$

This result coincides with intuition — at least when the intuition is well trained: The gradient is obtained from the gradient of  $E$  in the ambient space (which is  $e_2$ ), by a projection onto the tangential space.

Without loss of generality, we can write the solution  $\rho = \rho(t)$  of (2.1) as  $\rho(t) = (\cos(\psi(t)), \sin(\psi(t)))$ . This is true since every point on the manifold can be written as  $(\cos(\psi), \sin(\psi))$  for some  $\psi \in \mathbb{R}$ . We note in passing:  $\psi$  is called the lifting of  $\rho$  (Deutsch: “Liftung”).

With this notation, the gradient flow equation (2.1) reads

$$(2.6) \quad \psi'(t) (-\sin(\psi(t)), \cos(\psi(t))) = -\cos(\psi(t)) (-\sin(\psi(t)), \cos(\psi(t))).$$

The equation for  $\psi$  is therefore

$$(2.7) \quad \psi'(t) = -\cos(\psi(t)).$$

**2.2. A manifold that is given by a submersion.** In [1], the relevant manifold for the gradient flow is not given as a submanifold of a Hilbert-space or Banach-space. Instead, the Riemannian manifold is “parametrized” with a submersion. We want to illustrate also this concept with a simple example. We use a flat manifold  $\mathcal{M}^*$  and regard  $\mathcal{M}$  as the image of  $\mathcal{M}^*$  under some submersion  $\Phi$ . We note that, in this outline, it is not relevant that  $\mathcal{M}^*$  is flat. Our interest here is not the gradient flow, but the “right” Riemannian metric of  $\mathcal{M}$ .

A *submersion* is a differentiable map whose differential is everywhere surjective (while, for an *immersion*, the differential is everywhere injective).

Manifold defined by a submersion	Our example
flat manifold $\mathcal{M}^*$	$\mathcal{M}^* = \mathbb{R}^3$
elements $\Phi \in \mathcal{M}^*$	$\Phi = (\Phi_1, \Phi_2, \Phi_3)$
submersion $\Pi : \mathcal{M}^* \rightarrow \mathcal{M}$	$\Pi : \Phi \mapsto (\cos(\Phi_1), \sin(\Phi_1))$
tangent space $T_\Phi \mathcal{M}^*$	$T_\Phi \mathbb{R}^3 = \mathbb{R}^3$
metric tensor $g_\Phi^*(v_1, v_2)$	$g_\Phi^*(v_1, v_2) = \langle v_1, v_2 \rangle_{\mathbb{R}^3}$
tangential $T_\Phi \Pi : T_\Phi \mathcal{M}^* \rightarrow T_\rho \mathcal{M}$ where $\rho = \Pi(\Phi)$	$T_\Phi \Pi \langle (\lambda_1, \lambda_2, \lambda_3) \rangle = \lambda_1 (-\sin(\Phi_1), \cos(\Phi_1))$ where $\rho = (\cos(\Phi_1), \sin(\Phi_1))$
kernel $\ker T_\Phi \Pi$	$\ker T_\Phi \Pi = \{0\} \times \mathbb{R}^2$

TABLE 2. Abstract submersion and the example. Our example continues, we still consider  $\mathcal{M} = S^1 \subset \mathbb{R}^2$ .

**Definition 2.2** (Isometric submersion). *The submersion is an isometric submersion if the following relation holds between  $g^*$  and  $g$  (base points are connected via  $\rho = \Pi(\Phi)$ ):*

$$(2.8) \quad g_\rho(s, s) = \inf \{g_\Phi^*(v, v) \mid T_\Phi \Pi \langle v \rangle = s\}.$$

Let us check this property in our example: Vectors  $v \in T_\Phi \mathcal{M}^*$  are vectors  $v = (v_1, v_2, v_3) \in \mathbb{R}^3$ . Vectors  $s \in T_\rho \mathcal{M}$  are of the form  $s = \lambda (-\sin(\Phi_1), \cos(\Phi_1))$ . The condition  $T_\Phi \Pi \langle v \rangle = s$  reads  $v_1 (-\sin(\Phi_1), \cos(\Phi_1)) = \lambda (-\sin(\Phi_1), \cos(\Phi_1))$ , which is equivalent to  $v_1 = \lambda$ . Relation (2.8) therefore demands for every  $s = \lambda (-\sin(\Phi_1), \cos(\Phi_1))$ :

$$(2.9) \quad \lambda^2 = g_\rho(s, s) \stackrel{!}{=} \inf \{g_\Phi^*(v, v) \mid T_\Phi \Pi \langle v \rangle = s\} = \inf \{|v|_{\mathbb{R}^3}^2 \mid v_1 = \lambda\} = \lambda^2.$$

We see that equality holds for all  $\lambda \in \mathbb{R}$ . Therefore, in our example, the map  $\Pi$  is indeed an isometric submersion.

## REFERENCES

- [1] F. Otto. The geometry of dissipative evolution equations: the porous medium equation. *Comm. Partial Differential Equations*, 26(1-2):101–174, 2001.