Lecture notes: Existence result for Maxwell's equations on bounded Lipschitz domains

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Abstract: We give a proof for the existence of weak solutions to time-harmonic Maxwell's equations on bounded Lipschitz domains. This result is well-known; our aim here is to present, in a concise way, the arguments that lead from a compactness result for divergence free functions and a Helmholtz decomposition to the existence result.

1. Introduction

In this note, we study time-harmonic Maxwell's equations. Given is a bounded domain $\Omega \subset \mathbb{R}^3$, two coefficient functions $\varepsilon \in L^\infty(\Omega, \mathbb{R})$ and $\mu \in L^\infty(\Omega, \mathbb{R})$, a frequency $\omega > 0$ and right-hand sides $f_h, f_e \colon \Omega \to \mathbb{C}^3$. We seek for functions $E, H \colon \Omega \to \mathbb{C}^3$ that satisfy

$$\operatorname{curl} E = i\omega\mu H + f_h \quad \text{in } \Omega\,, \tag{1.1a}$$

$$\operatorname{curl} H = -i\omega\varepsilon E + f_e \quad \text{in } \Omega\,, \tag{1.1b}$$

with a homogeneous tangential boundary condition for E,

$$E \times \nu = 0 \quad \text{on } \partial\Omega\,, \tag{1.1c}$$

where ν is the exterior normal vector on $\partial\Omega$. The boundary condition (1.1c) is understood as $E \in H_0(\operatorname{curl}, \Omega)$. The weak solution concept and the spaces $H_0(\operatorname{curl}, \Omega)$ and $H(\operatorname{curl}, \Omega)$ are defined below in (1.2)–(1.4).

1.1. Main result.

We present and prove a result on the existence of solutions to the Maxwell system (1.1). We will actually provide two different proofs, one with Fredholm operator theory, one with a limiting absorption principle. We emphasize that the result is classical, [3] is one of the more recent references. Our aim here is to present a short derivation.

Theorem 1.1 (Existence and uniqueness). *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and $\omega > 0$ be a frequency. Let $\varepsilon \in L^\infty(\Omega, \mathbb{R})$ and $\mu \in L^\infty(\Omega, \mathbb{R})$ be coefficient functions such that, for some $c_0 > 0$, there holds $\varepsilon \geq c_0$ and $\mu \geq c_0$. We assume that system (1.1) with $f_h = 0$ and $f_e = 0$ has only the trivial solution.*

Then, for every $f_h, f_e \in L^2(\Omega, \mathbb{C}^3)$, system (1.1) has a unique weak solution $(E, H) \in H_0(\operatorname{curl}, \Omega) \times H(\operatorname{curl}, \Omega)$.

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[2]This version coincides in principle with the original version from 2024, we corrected some inconsistencies and typos.

We note that the uniqueness follows immediately from the linearity of the Maxwell system and the assumption of uniqueness for $f_h = f_e = 0$.

In order to keep this exposition as simple as possible, we assume $\operatorname{div}(f_h) = 0$ in Ω and $f_h \cdot \nu = 0$ on $\partial\Omega$. This is understood in the weak sense, $\int_\Omega f_h \cdot \nabla\psi = 0$ for every $\psi \in H^1(\Omega)$. With the help of the Helmholtz decomposition, it can be shown that this assumption is not a loss of generality.

1.2. **Weak form.** To formulate (1.1) in a weak sense, we use the space of L^2-functions such that the distributional curl of the function is again of class L^2. More precisely, we use

$$H(\operatorname{curl}, \Omega) := \left\{ u \in L^2(\Omega, \mathbb{C}^3) \mid \operatorname{curl} u \in L^2(\Omega, \mathbb{C}^3) \right\} . \tag{1.2}$$

This is a Hilbert space with the norm $\|u\|^2_{H(\operatorname{curl},\Omega)} := \int_\Omega \{|u|^2 + |\operatorname{curl} u|^2\}$ and the scalar product $\langle u, \varphi\rangle := \langle u, \varphi\rangle_{L^2(\Omega)} + \langle \operatorname{curl} u, \operatorname{curl} \varphi\rangle_{L^2(\Omega)}$.

Let us motivate a weak formulation of the Maxwell system. We choose $\phi \in H(\operatorname{curl}, \Omega)$ arbitrarily and use $\varepsilon^{-1} \operatorname{curl} \phi$ as a test-function for (1.1b). On the right-hand side occurs an integral over $E \cdot \operatorname{curl} \phi$, which is identical to the integral over $\operatorname{curl} E \cdot \phi$ because of the boundary condition (1.1c). Replacing $\operatorname{curl} E$ with the expression of (1.1a), we find

$$\int_\Omega \left\{ \varepsilon^{-1} \operatorname{curl} H \cdot \operatorname{curl} \phi - \omega^2 \mu H \cdot \phi \right\} = \int_\Omega \left\{ -i\omega f_h \cdot \phi + \varepsilon^{-1} f_e \cdot \operatorname{curl} \phi \right\} \quad \forall\, \phi\,, \tag{1.3}$$

where the test-functions are all $\phi \in H(\operatorname{curl}, \Omega)$. We choose (1.3) as the weak form of system (1.1).

Note that, once a solution H of (1.3) is found, E can be defined with formula (1.1b) and the pair (E, H) solves the coupled system. The tangential boundary condition $E \times \nu = 0$ on $\partial\Omega$ is satisfied in the form $E \in H_0(\operatorname{curl}, \Omega)$, where

$$H_0(\operatorname{curl}, \Omega) := \left\{ u \in H(\operatorname{curl}, \Omega) \middle| \int_\Omega \operatorname{curl}\ u \cdot \eta = \int_\Omega u \cdot \operatorname{curl}\ \eta \quad \forall \eta \in H(\operatorname{curl}, \Omega) \right\} . \tag{1.4}$$

Our aim is to prove Theorem 1.1 with the help of the compactness result of Lemma 1.2 and with the Helmholtz decomposition of Lemma 1.3.

1.3. **Tool I: Compactness.** The following lemma is classical in the theory of spaces of functions with curl in L^2.

Lemma 1.2 (Compactness)**.** *Let Ω be a bounded Lipschitz domain and let $\mu \in L^\infty(\Omega)$ be bounded below by some positive constant. Then the space*

$$Y := \left\{ u \in H(\operatorname{curl}, \Omega) \middle| \int_\Omega \mu\, u \cdot \nabla\psi = 0 \textit{ for all } \psi \in H^1(\Omega) \right\} \tag{1.5}$$

is compactly embedded in $L^2(\Omega, \mathbb{C}^3)$.

Only standard methods are necessary to prove Lemma 1.2. In order to make that clear, we give a proof in Section 4. Proposition 4.1 in Section 4 is even more general in the sense that only the L^2-control of the divergence of μu is demanded (and not that the divergence vanishes).

Comments on the literature regarding Lemma 1.2. The lemma is stated in [5, Lemma A.1] with our assumption on the coefficient. A very similar version is [3, Corollary 4.36], where positive definite symmetric matrix valued coefficients $\mu \in$

$L^\infty(\Omega, \mathbb{C}^{3\times 3})$ are allowed. We remark that closely related results are also stated and proved for the subset of $H_0(\mathrm{curl}, \Omega)$ in [3, Theorem 4.24] and [6, Theorem 4.7], the latter with additional regularity assumptions on the coefficient.

1.4. **Tool II: Helmholtz decomposition.** The space Y of (1.5) is complemented with the space of gradients,

$$G := \left\{ u \in H(\mathrm{curl}, \Omega) \,\middle|\, \exists \psi \in H^1(\Omega) \colon u = \nabla\psi \right\} . \tag{1.6}$$

The choice is such that Y is orthogonal to G in the space $L^2(\Omega, \mathbb{C}^3)$ with the weighted scalar product $\langle u, v\rangle = \int_\Omega \mu\, u \cdot \overline{v}$. Furthermore, since the (distributional) curl of a gradient always vanishes, $\mathrm{curl}(\nabla\psi) = 0$, the two subspaces are also orthogonal in $X := H(\mathrm{curl}, \Omega)$ with the scalar product $\langle u, v\rangle_X := \int_\Omega \{\mu\, u \cdot \overline{v} + \mathrm{curl}\, u \cdot \mathrm{curl}\, \overline{v}\}$. By construction, Y is the $\langle .,. \rangle_X$-orthogonal complement of G, we may write this as $Y = G^\perp$. We therefore have the following result:

Lemma 1.3 (Helmholtz decomposition)**.** *The space $X := H(\mathrm{curl}, \Omega)$ has the orthogonal decomposition $X = Y \oplus G$. In particular, an arbitrary element $u \in X$ can be written uniquely as $u = v + \nabla\psi$ with $v \in Y$ and $\psi \in H^1(\Omega)$.*

2. Existence proof with Fredholm operator theory

In our first existence proof, we re-formulate problem (1.3) with the Helmholtz decomposition. We seek solutions H of (1.3) in the space Y of (1.5). We recall that the condition $H \in Y$ encodes that the divergence of μH vanishes and that $H \cdot \nu$ vanishes along the boundary.

Lemma 2.1 (Equivalent formulation in Y)**.** *The Maxwell system in the form (1.3) is equivalent to: Find $H \in Y$ with*

$$\int_\Omega \left\{\varepsilon^{-1}\, \mathrm{curl}\, H \cdot \mathrm{curl}\, \varphi - \omega^2 \mu H \cdot \varphi\right\} = \int_\Omega \left\{-i\omega f_h \cdot \varphi + \varepsilon^{-1} f_e \cdot \mathrm{curl}\, \varphi\right\} \quad \forall\, \varphi \in Y . \tag{2.1}$$

Proof. Let H be a solution of (1.3). For arbitrary $\psi \in H^1(\Omega, \mathbb{R})$, we define $\varphi = \nabla\psi$; because of the distributional equality $\mathrm{curl}\, \nabla\psi = 0$, there holds $\varphi \in H(\mathrm{curl}, \Omega)$. Using φ as a test-function in (1.3), we note that all terms except for $\int_\Omega \omega^2 \mu H \cdot \varphi$ vanish because of $\mathrm{curl}\, \varphi = 0$ and $\int_\Omega f_h \cdot \nabla\psi = 0$ for $\psi \in H^1(\Omega)$. The result is that μH is orthogonal to gradients, i.e., $H \in Y$. Since the space of test-functions is smaller in (2.1) than in (1.3), this implies that (2.1) holds.

Vice versa, let H be a solution of (2.1). Given an arbitrary test-function $\phi \in X = H(\mathrm{curl}, \Omega)$, we write $\phi = \varphi + \nabla\psi$ with $\varphi \in Y$ and $\psi \in H^1(\Omega)$, see Lemma 1.3. We use that (1.3) is linear in ϕ, we can treat all contributions separately. Inserting $\nabla\psi$ in (1.3), arguing as above and exploiting $H \in Y$, we find that all terms vanish. Inserting φ, the equality holds because of (2.1). This shows that (1.3) is satisfied for the test-function ϕ. Since ϕ was arbitrary, H is a solution of (1.3). □

The re-formulation (2.1) is useful since Lemma 1.2 provides compactness of Y.

Proof of Theorem 1.1 with Fredholm operators. On Y, we define sesquilinear forms $a, b \colon Y \times Y \to \mathbb{C}$ and a linear right-hand side $f \colon Y \to \mathbb{C}$ by setting, for every

$u, \phi \in Y$,

$$a(u,\phi) := \int_\Omega \left\{ u \cdot \overline{\phi} + \varepsilon^{-1} \operatorname{curl} u \cdot \operatorname{curl} \overline{\phi} \right\} ,$$

$$b(u,\phi) := \int_\Omega \left\{ u \cdot \overline{\phi} + \omega^2 \mu\, u \cdot \overline{\phi} \right\} , \quad f(\phi) := \int_\Omega \left\{ -i\omega f_h \cdot \overline{\phi} + \varepsilon^{-1} f_e \cdot \operatorname{curl} \overline{\phi} \right\} .$$

Lemma 2.1 yields: The Maxwell system (1.3) is equivalent to: Find $H \in Y$ with

$$a(H,\varphi) - b(H,\varphi) = f(\varphi) \qquad \forall\, \varphi \in Y . \tag{2.2}$$

This is the equation that we have to solve.

The sesquilinear form a defines a map $A\colon Y \to Y'$ from Y into the dual space Y' by $Au := a(u,\cdot)$. By definition of the scalar product in $Y \subset X = H(\operatorname{curl},\Omega)$, the form a is coercive on Y. The Lemma of Lax–Milgram implies that $A\colon Y \to Y'$ is invertible.

We now exploit that the embedding $\iota\colon Y \to L^2(\Omega,\mathbb{C}^3)$ is compact, see Lemma 1.2. In the calculations below, we suppress the embedding $L^2(\Omega,\mathbb{C}^3) \to Y'$. The multiplication map $B\colon u \mapsto (1+\omega^2\mu)\, u$ corresponding to b is linear and bounded as a map $B\colon L^2(\Omega,\mathbb{C}^3) \to L^2(\Omega,\mathbb{C}^3)$. We denote the concatenation with an embedding into Y' with the same letter, $B\colon L^2(\Omega,\mathbb{C}^3) \to Y'$.

The field $H \in Y$ solves (2.2) if and only if

$$A\,H - (B \circ \iota)\,H = f \qquad \text{in } Y' .$$

Applying A^{-1}, we find the equivalent relation

$$(\operatorname{id} - A^{-1} \circ B \circ \iota) H = A^{-1} f \qquad \text{in } Y .$$

The operator $A^{-1} \circ B \circ \iota$ is compact, since ι is compact and the other operators are continuous. This implies that the operator $F := \operatorname{id} - A^{-1} \circ B \circ \iota$ is a Fredholm operator of index zero, see [1, Theorem 11.8].

One assumption in Theorem 1.1 is that the homogeneous system has only the trivial solution. Since all our re-formulations of the Maxwell system are equivalent, this implies that $F = \operatorname{id} - A^{-1} \circ B \circ \iota$ is injective. Since the index is zero, the injectivity implies the surjectivity. Surjectivity of F implies that for every right-hand side there exists a solution of (2.2). Since the formulations are equivalent, we have also found a solution of (1.1) and Theorem 1.1 is proved. $\square$

3. Existence proof with a limiting absorption principle

In this section, we prove Theorem 1.1 with a limiting absorption principle. A limiting absorption principle considers the equation with a damping term, which is sent to zero. In this text, we use a small constant $\delta > 0$ and replace the pre-factor $\omega^2\mu$ in equation (1.3) by $\omega^2\mu - i\delta$. This equation has a solution H_δ. We find a limit $H = \lim_{\delta\to 0} H_\delta$, which is a solution of (1.3).

In the first step, we seek a solution $H_\delta \in H(\operatorname{curl},\Omega)$ of

$$\int_\Omega \left\{ \varepsilon^{-1} \operatorname{curl} H_\delta \cdot \operatorname{curl} \phi - (\omega^2\mu - i\delta) H_\delta \cdot \phi \right\} = \int_\Omega \left\{ -i\omega f_h \cdot \phi + \varepsilon^{-1} f_e \cdot \operatorname{curl} \phi \right\} \quad \forall\, \phi , \tag{3.1}$$

where the test-functions are arbitrary functions $\phi \in H(\operatorname{curl},\Omega)$.

Proof of Theorem 1.1 with limiting absorption. Lemma 3.1 provides a unique solution H_δ of (3.1). Lemma 3.2 shows that the sequence H_δ is bounded in $H(\operatorname{curl},\Omega)$ (in the setting of Theorem 1.1, where it is assumed that (1.3) has only the trivial solution for $f_h = f_e = 0$). Since $H(\operatorname{curl},\Omega)$ is reflexive, there exists a subsequence of H_δ, which converges weakly to some limit $H \in H(\operatorname{curl},\Omega)$. The weak convergence allows us to perform the limit $\delta \to 0$ in (3.1) along the subsequence. We obtain that H solves (1.3). □

Lemma 3.1 (Existence of a solution for the problem with absorption)**.** *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and $\omega \in \mathbb{R} \setminus \{0\}$ a frequency. Let $\varepsilon \in L^\infty(\Omega,\mathbb{R})$ and $\mu \in L^\infty(\Omega,\mathbb{R})$ be coefficient functions such that, for some $c_0 > 0$, there holds $\varepsilon \geq c_0$ and $\mu \geq c_0$. Then there exists $\delta_0 > 0$ such that, for every $\delta \in (0,\delta_0)$, equation (3.1) has a unique weak solution $H_\delta \in H(\operatorname{curl},\Omega)$.*

Proof. We define the sesquilinear form a_δ on $H(\operatorname{curl},\Omega)$ by setting, for $u,\varphi \in H(\operatorname{curl},\Omega)$,

$$a_\delta(u,\varphi) := \int_\Omega \left\{\varepsilon^{-1}\operatorname{curl} u \cdot \operatorname{curl}\overline{\varphi} + (i\delta - \omega^2\mu)u\cdot\overline{\varphi}\right\} .$$

The form a_δ allows us to rewrite (3.1) as

$$a_\delta(H_\delta,\phi) = \int_\Omega \left\{-i\omega f_h \cdot \overline{\phi} + \varepsilon^{-1} f_e \cdot \operatorname{curl}\overline{\phi}\right\} \qquad \forall\, \phi \in H(\operatorname{curl},\Omega) . \tag{3.2}$$

We calculate, for arbitrary $u \in H(\operatorname{curl},\Omega)$,

$$\begin{aligned}&\operatorname{Im} a_\delta(u,u) = \delta\|u\|^2_{L^2(\Omega)} ,\\ &\operatorname{Re} a_\delta(u,u) \geq \|\varepsilon\|^{-1}_{L^\infty}\,\|\operatorname{curl} u\|^2_{L^2(\Omega)} - \omega^2\|\mu\|_{L^\infty}\,\|u\|^2_{L^2(\Omega)} .\end{aligned}$$

We observe the following fact in $\mathbb{R}^2 \equiv \mathbb{C}$: For every vector $x = (x_1,x_2) \in \mathbb{R}^2$ and every $s \in [0,1]$, there holds $|x| \geq \max\{|x_1|,|x_2|\} \geq (1-s)|x_1| + s|x_2|$. This inequality allows to calculate, with $s = \delta^2$,

$$\begin{aligned}|a_\delta(u,u)| &\geq (1-\delta^2)\operatorname{Im} a_\delta(u,u) + \delta^2 \operatorname{Re} a_\delta(u,u)\\ &\geq (1-\delta^2)\delta\|u\|^2_{L^2(\Omega)} + \delta^2\left(\|\varepsilon\|^{-1}_{L^\infty}\,\|\operatorname{curl} u\|^2_{L^2(\Omega)} - \omega^2\|\mu\|_{L^\infty}\,\|u\|^2_{L^2(\Omega)}\right) .\end{aligned}$$

Choosing $\delta_0 > 0$ small, we achieve $(1-\delta^2)\delta \geq 2\delta^2\omega^2\|\mu\|_{L^\infty}$ for all $\delta < \delta_0$, and obtain that a_δ is coercive. Problem (3.2) for H_δ can therefore be solved with the Lemma of Lax–Milgram. □

Lemma 3.2 (Boundedness of solutions to the problem with absorption)**.** *Let the assumptions of Lemma 3.1 be satisfied. For a sequence $\delta \to 0$, let $H_\delta \in H(\operatorname{curl},\Omega)$ be the corresponding sequence of solutions of (3.1). Suppose that (1.3) for $f_h = f_e = 0$ has only the trivial solution $H = 0$. Then, H_δ is bounded in $H(\operatorname{curl},\Omega)$.*

Proof. Step 1: Preparation. For a contradiction argument, we assume that there exists a subsequence H_δ such that $\|H_\delta\|_{H(\operatorname{curl},\Omega)} \to \infty$. We use H_δ as a test-function in (3.2) to obtain

$$\|\varepsilon\|_{L^\infty}^{-1} \|\operatorname{curl} H_\delta\|_{L^2(\Omega)}^2 \le \int_\Omega \varepsilon^{-1} \operatorname{curl} H_\delta \cdot \operatorname{curl} \overline{H}_\delta$$
$$= \int_\Omega -(i\delta - \omega^2\mu) H_\delta \cdot \overline{H_\delta} + \int_\Omega \left\{ -i\omega f_h \cdot \overline{H_\delta} + \varepsilon^{-1} f_e \cdot \operatorname{curl} \overline{H_\delta} \right\} .$$

For arbitrary $\lambda > 0$, we continue this calculation with Young's inequality and find

$$\|\varepsilon\|_{L^\infty}^{-1} \|\operatorname{curl} H_\delta\|_{L^2(\Omega)}^2 \le C \|H_\delta\|_{L^2(\Omega)}^2 + C_\lambda + \lambda \|\operatorname{curl} H_\delta\|_{L^2(\Omega)}^2 ,$$

for some C depending on ω, f_h and μ, and C_λ depending on ε, f_e and λ. Choosing $\lambda = \|\varepsilon\|_{L^\infty}^{-1}/2$ and subtracting the term $\lambda \|\operatorname{curl} H_\delta\|_{L^2(\Omega)}^2$ on both sides, we find, for some constant C, the inequality $\|\operatorname{curl} H_\delta\|_{L^2(\Omega)}^2 \le C\,(1 + \|H_\delta\|_{L^2(\Omega)}^2)$.

In particular: The divergence $\|H_\delta\|_{H(\operatorname{curl},\Omega)} \to \infty$ implies $\|H_\delta\|_{L^2(\Omega)} \to \infty$.

Step 2: Normalization. We normalize the sequence H_δ and consider the new sequence $\tilde H_\delta := H_\delta / \|H_\delta\|_{L^2(\Omega,\mathbb{C}^3)}$. This sequence satisfies

$$a_\delta(\tilde H_\delta, \phi) = \|H_\delta\|_{L^2(\Omega,\mathbb{C}^3)}^{-1} \int_\Omega \left\{ -i\omega f_h \cdot \overline{\phi} + \varepsilon^{-1} f_e \cdot \operatorname{curl} \overline{\phi} \right\} \quad \forall\, \phi \in H(\operatorname{curl}, \Omega) . \tag{3.3}$$

The result of Step 1 together with $\|\tilde H_\delta\|_{L^2(\Omega,\mathbb{C}^3)} = 1$ shows that $\|\operatorname{curl} \tilde H_\delta\|_{L^2(\Omega)}^2$ is bounded.

Since $H(\operatorname{curl}, \Omega)$ is reflexive, there exists a subsequence of $\tilde H_\delta$, which converges weakly to some limit $\tilde H \in H(\operatorname{curl}, \Omega)$, in particular $\tilde H_\delta \rightharpoonup \tilde H$ in $L^2(\Omega)$ and $\operatorname{curl} \tilde H_\delta \rightharpoonup \operatorname{curl} \tilde H$ in $L^2(\Omega)$. The weak convergence allows us to perform the limit $\delta \to 0$ in (3.3) along the subsequence. We obtain that the limit $\tilde H$ solves (1.3) for $f_h = f_e = 0$. Our assumption was that there is no non-trivial solution to the homogeneous problem; this implies $\tilde H = 0$.

Step 3: Strong convergence. In this step we show the strong convergence of $\tilde H_\delta$ in $L^2(\Omega, \mathbb{C}^3)$ along a subsequence. Once this is obtained, we have the desired contradiction: $\|\tilde H_\delta\|_{L^2(\Omega)} = 1$ is in conflict with the strong convergence $\tilde H_\delta \to \tilde H = 0$.

In order to show the strong convergence, we decompose $\tilde H_\delta$ by means of the Helmholtz decomposition of Lemma 1.3: $\tilde H_\delta = \tilde H_\delta^Y + \nabla \psi_\delta$ for $\tilde H_\delta^Y \in Y$ and $\psi_\delta \in H^1(\Omega)$. The compactness of Y shown in Lemma 1.2 allows us to pass to a subsequence such that $\tilde H_\delta^Y$ converges strongly in $L^2(\Omega, \mathbb{C}^3)$.

Let us use the test-function $\phi = \nabla \overline{\psi_\delta}$ in (3.1). Since the curl of a gradient vanishes and since we assumed that f_h is orthogonal to gradients, we obtain

$$\int_\Omega (\mu - i\omega^{-2}\delta) \tilde H_\delta \cdot \nabla \overline{\psi_\delta} = 0 .$$

Inserting the decomposition $\tilde H_\delta = \tilde H_\delta^Y + \nabla \psi_\delta$, we find

$$i \int_\Omega \omega^{-2} \delta \tilde H_\delta \cdot \nabla \overline{\psi_\delta} - \int_\Omega \mu |\nabla \psi_\delta|^2 = \int_\Omega \mu \tilde H_\delta^Y \cdot \nabla \overline{\psi_\delta} = 0 , \tag{3.4}$$

where we used the property $\tilde H_\delta^Y \in Y$ in the last equality. The lower bound $\mu \ge c_0 > 0$ allows us to obtain from (3.4)

$$c_0 \|\nabla \psi_\delta\|_{L^2(\Omega)}^2 \le \int_\Omega \mu |\nabla \psi_\delta|^2 = i \int_\Omega \omega^{-2} \delta \tilde H_\delta \cdot \nabla \overline{\psi_\delta} .$$

The Cauchy-Schwarz inequality and the normalization $\|\tilde{H}_\delta\|_{L^2(\Omega)} = 1$ imply

$$c_0\|\nabla\psi_\delta\|^2_{L^2(\Omega)} \le \delta\omega^{-2}\|\tilde{H}_\delta\|_{L^2(\Omega)}\|\nabla\psi_\delta\|_{L^2(\Omega)} = \delta\omega^{-2}\|\nabla\psi_\delta\|_{L^2(\Omega)}\,.$$

Dividing by $\|\nabla\psi_\delta\|_{L^2(\Omega)}$ if this term is different from zero, we find $\|\nabla\psi_\delta\|_{L^2(\Omega)} \to 0$ as $\delta \to 0$. This is the desired strong convergence of $\nabla\psi_\delta$.

The strong convergence of $\tilde{H}^Y_\delta$ together with the strong convergence of $\nabla\psi_\delta$ implies the strong convergence of $\tilde{H}_\delta = \tilde{H}^Y_\delta + \nabla\psi_\delta$ in $L^2(\Omega)$. This provides the desired contradiction and concludes the proof. □

4. Compactness property

We include a proof for the compactness result in order to have this exposition self-contained. The compactness has some relations with the div-curl lemma, sometimes called compensated compactness. Knowledge on the curl and on the divergence of a function somehow controls all derivatives. Loosely speaking, this property is already suggested by the relation $\Delta = -\operatorname{curl}\operatorname{curl} + \nabla\operatorname{div}$.

Proposition 4.1 (Compactness when rotation and divergence are bounded in L^2)**.** *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and $\mu \in L^\infty(\Omega,\mathbb{R})$ uniformly bounded away from zero, i.e. $\mu \ge c_0$ for a constant $c_0 > 0$, let $u_j \in L^2(\Omega,\mathbb{C}^3)$ be a bounded sequence such that the distributional curl of u_j and the distributional divergence of μu_j are bounded in L^2,*

$$\operatorname{curl} u_j =: f_j \in L^2(\Omega,\mathbb{C}^3) \text{ bounded,} \tag{4.1}$$

$$\operatorname{div}(\mu u_j) =: g_j \in L^2(\Omega,\mathbb{C}) \text{ bounded.} \tag{4.2}$$

We additionally demand that normal boundary values of u_j are vanishing by imposing $\int_\Omega \mu u_j\cdot\nabla\varphi = -\int_\Omega g_j\,\varphi$ for all $\varphi \in H^1(\Omega,\mathbb{C})$.

Under these assumptions, there exists a subsequence $j \to \infty$ and a limit function $u \in L^2(\Omega,\mathbb{C}^3)$ such that $u_j \to u$ in $L^2(\Omega,\mathbb{C}^3)$ along this subsequence.

Proof. We proceed as follows: In Steps 1 to 3, we prove the Proposition under the additional assumptions that Ω is simply connected, $\mathbb{R}^3\setminus\overline{\Omega}$ is connected and that the factor is $\mu \equiv 1$. In Step 4, we treat general coefficients μ. In Step 5, we treat general domains. Since Ω is bounded, it is contained in a ball; we choose $R > 0$ with $\overline{\Omega} \subset \tilde{\Omega} := B_R(0) \subset \mathbb{R}^3$. Later on, we will choose a smaller open bounded set $\tilde{\Omega} \subset \mathbb{R}^3$. The divergence of u_j is trivially extended, we define $\tilde{g}_j \in L^2(\tilde{\Omega})$ by setting $\tilde{g}_j(x) := g_j(x)$ for $x \in \Omega$ and $\tilde{g}_j(x) := 0$ for $x \in \tilde{\Omega}\setminus\Omega$.

Step 1: A vector potential for f_j. The function f_j has a vanishing divergence in Ω. We extend it to a function $\tilde{f}_j$ such that $\operatorname{div}\tilde{f}_j$ vanishes in $\tilde{\Omega}$. The existence of such an extension (with bounds for the norm) is verified below in Lemma 4.2.

As a function with vanishing divergence in a ball, $\tilde{f}_j$ can be written with a vector potential $\tilde{w}_j \in L^2(\tilde{\Omega},\mathbb{C}^3)$,

$$\tilde{f}_j = \operatorname{curl}\tilde{w}_j\,. \tag{4.3}$$

The existence of such a vector potential is classical theory, $\tilde{w}_j$ can be constructed with the help of path integrals over the components of $\tilde{f}_j$, see [2] and [7] for modern references. The vector potentials $\tilde{w}_j$ can be chosen so that the sequence is bounded in $L^2(\tilde{\Omega})$.

Step 2: Modification of the vector potential $\tilde{w}_j$. We want to modify the vector potential such that it is divergence-free. We find such a modified potential in the form $\tilde{W}_j = \tilde{w}_j - \nabla\tilde{\psi}_j$ where $\tilde{\psi}_j$ solves

$$\Delta\tilde{\psi}_j = \operatorname{div}\tilde{w}_j$$

in $\tilde{\Omega}$ (with homogeneous Dirichlet conditions on $\partial\tilde{\Omega}$) so that $\operatorname{div}(\tilde{W}_j) = 0$. Since $\tilde{w}_j$ is bounded in $L^2(\tilde{\Omega})$, we find $\tilde{\psi}_j$ bounded in $H^1(\tilde{\Omega})$. The result is a bounded sequence $\tilde{W}_j \in L^2(\tilde{\Omega}, \mathbb{C}^3)$ with the (distributional) rotation $\tilde{f}_j$ and vanishing divergence. We note that, because of $\Delta = -\operatorname{curl}\operatorname{curl} + \nabla\operatorname{div}$, the function $\tilde{W}_j \in H^1(\tilde{\Omega}, \mathbb{C}^3)$ satisfies, in the sense of distributions, $\Delta\tilde{W}_j = -\operatorname{curl}\tilde{f}_j \in H^{-1}(\tilde{\Omega})$. The sequence $\tilde{W}_j$ is therefore locally of class H^1. By possibly choosing a smaller open set $\tilde{\Omega}$ (still containing $\overline{\Omega}$), we achieve that the sequence $\tilde{W}_j \in H^1(\tilde{\Omega})$ is bounded (Caccioppoli's inequality). The restriction $W_j := \tilde{W}_j|_\Omega$ has the rotation f_j and a vanishing divergence.

Since the sequence is bounded in $H^1(\tilde{\Omega}, \mathbb{C}^3)$, there exists a subsequence with strong convergence of $\tilde{W}_j$ in $L^2(\tilde{\Omega}, \mathbb{C}^3)$. We consider only this subsequence in the following.

Step 3: A scalar potential. We consider the function $v_j := u_j - W_j \in L^2(\Omega, \mathbb{R}^3)$ with vanishing rotation. Since Ω is simply connected, the function has a scalar potential ϕ_j,

$$v_j = \nabla\phi_j\,.$$

We have therefore written the original sequence as $u_j = W_j + \nabla\phi_j$. Since W_j converges strongly, the lemma is proven when we show that $\nabla\phi_j$ converges strongly in L^2 along a subsequence.

With the normal vector ν on $\partial\Omega$, the function ϕ_j is a solution of

$$\Delta\phi_j = g_j \text{ in } \Omega\,, \qquad \nu\cdot\nabla\phi_j = -\nu\cdot W_j \text{ on } \partial\Omega\,,$$

where we used $\nu\cdot u_j = 0$ on the boundary. This must be understood in the weak form: The potential ϕ_j is a solution of

$$\int_\Omega \nabla\phi_j\cdot\nabla\varphi = -\int_\Omega g_j\,\varphi - \int_{\partial\Omega}\nu\cdot W_j\,\varphi \tag{4.4}$$

for every $\varphi \in H^1(\Omega)$. To have uniqueness of solutions, we demand that the integral condition $\int_\Omega \phi_j = 0$ and define $H^1_*(\Omega)$ as the space of $H^1(\Omega)$-functions satisfying this condition. The integrability condition on g_j and W_j is that the right-hand side of (4.4) vanishes for constant functions φ. In our setting, both integrals vanish for constant φ: The first because of our assumption on u_j, the second because of $\operatorname{div}W_j = 0$.

The theorem of Lax–Milgram can be applied, the solution operator to this problem, $\mathcal{T}_0\colon (g_j, \nu\cdot W_j) \mapsto \phi_j$, is a bounded linear operator $\mathcal{T}_0\colon (H^1(\Omega))' \times L^2(\partial\Omega) \to H^1_*(\Omega)$ (we suppress here that the argument must satisfy the integrability condition).

Additionally, we have the compactness of the embedding $L^2(\Omega) \subset (H^1(\Omega))'$ and the compactness of the trace operator $H^1(\Omega) \to L^2(\partial\Omega)$. This shows that the solution operator $\mathcal{T}_1\colon (g_j, W_j) \mapsto \phi_j$ is compact as a map $L^2(\Omega) \times H^1(\Omega) \to H^1_*(\Omega)$. Since g_j and W_j are bounded in the left spaces, we conclude that $\phi_j = \mathcal{T}_1(g_j, W_j)$ has a convergent subsequence in $H^1(\Omega)$. This shows that $\nabla\phi_j$ has a convergent subsequence in $L^2(\Omega, \mathbb{R}^3)$ and concludes the proof for simply connected domains and $\mu \equiv 1$.

Step 4: General coefficients μ. We now consider a general positive coefficient $\mu \in L^\infty(\Omega, \mathbb{R})$. In order to show compactness, we consider once more a bounded sequence u_j as above, now with general μ.

We claim that it is sufficient to consider the case $\operatorname{div}(\mu u_j) = 0$. Indeed, in the general case, we consider the sequence $\hat{u}_j := u_j - \nabla\xi_j$ where $\xi_j \in H^1(\Omega)$ with $\int_\Omega \xi_j = 0$ solves the homogeneous Neumann problem $\operatorname{div}(\mu\nabla\xi_j) = \operatorname{div}(\mu u_j)$ with $\nu \cdot \mu\nabla\xi_j|_{\partial\Omega} = 0$. For a subsequence, $\operatorname{div}(\mu u_j)$ is converging weakly in $L^2(\Omega)$ and hence strongly in $(H^1(\Omega))'$, which implies that ξ_j is strongly converging in $H^1(\Omega)$ and $\nabla\xi_j$ strongly in $L^2(\Omega)$. It is therefore sufficient to prove the strong convergence of $\hat{u}_j$ in $L^2(\Omega)$. This justifies the claim.

We use the Helmholtz decomposition $H(\operatorname{curl}, \Omega) = Y \oplus G$ with Y and G defined in (1.5) and (1.6), respectively. The decomposition was discussed in Lemma 1.3 for general μ, we write $H(\operatorname{curl}, \Omega) = Y_\mu \oplus G$ when we use the coefficient μ and, accordingly, $H(\operatorname{curl}, \Omega) = Y_1 \oplus G$ when we use the coefficient 1. The corresponding projections are bounded.

We decompose u_j with respect to $H(\operatorname{curl}, \Omega) = Y_1 \oplus G$ and write

$$u_j \;=\; v_j \;+\; \nabla\psi_j \quad \text{ with } v_j \in Y_1 \text{ and } \psi_j \in H^1(\Omega, \mathbb{C})\,.$$

Since u_j is bounded in $H(\operatorname{curl}, \Omega)$ and since projections are bounded, the sequence $v_j \in Y_1$ is bounded in $H(\operatorname{curl}, \Omega)$. Since it has a vanishing divergence, Steps 1–3 yield that there exists a subsequence, again denoted by v_j, which converges in $L^2(\Omega, \mathbb{C}^3)$. With this knowledge, we now read the previous decomposition in the form

$$v_j \;=\; u_j \;-\; \nabla\psi_j\,.$$

We note that this is an orthogonal decomposition of v_j in $L^2(\Omega, \mathbb{C}^3) = \tilde{Y}_\mu \oplus G$ with respect to the scalar product $\langle u, v\rangle = \int_\Omega \mu\, u \cdot \overline{v}$ where

$$\tilde{Y}_\mu := \left\{ u \in L^2(\Omega, \mathbb{C}^3) \,\middle|\, \int_\Omega \mu\, u \cdot \nabla\psi = 0 \text{ for all } \psi \in H^1(\Omega) \right\}.$$

The projections on G and $\tilde{Y}_\mu$ are continuous with respect to the L^2-norm. Since v_j converges in $L^2(\Omega, \mathbb{C}^3)$, we conclude that u_j converges in $L^2(\Omega, \mathbb{C}^3)$. This concludes the proof.

Step 5: Removing the assumption of simple connectedness. Let now Ω be an arbitrary bounded Lipschitz domain. We choose a finite family of simply connected Lipschitz subdomains $\Omega_k \subset \Omega$, $k = 1, ..., K$ that cover Ω. We choose a subordinate family of smooth cut-off functions η_k. The lemma is then applied in each subdomain Ω_k to the sequence $u_j\eta_k$. □

The task in the subsequent lemma is to extend the function F to a function $\tilde{F}$ such that the normal component of $\tilde{F} \cdot \nu$ has no jump on $\partial\Omega$. The principle idea is simple: We set $\tilde{F} = \nabla p$ outside Ω, where p is a harmonic function outside Ω and has the desired Neumann values on $\partial\Omega$. The main task is to perform these steps in the weak form.

Lemma 4.2 (Extension with bounded divergence)**.** *Let Ω and $\tilde{\Omega}$ be bounded Lipschitz domains in $\mathbb{R}^n$ with $\overline{\Omega} \subset \tilde{\Omega}$ with $\tilde{\Omega} \setminus \overline{\Omega}$ connected. Let $F \in L^2(\Omega, \mathbb{R}^n)$ be a function with a distributional divergence $\rho = \operatorname{div}(F) \in L^2(\Omega, \mathbb{R})$. We denote the trivial extension of ρ to $\tilde{\Omega}$ by $\tilde{\rho} : \tilde{\Omega} \to \mathbb{R}$. Then, there exists an extension $\tilde{F} \in L^2(\tilde{\Omega}, \mathbb{R}^n)$ with the distributional divergence $\tilde{\rho} = \operatorname{div}(\tilde{F})$. The extension can* *be chosen with bounded norm,* $\|\tilde F\|_{L^2(\tilde\Omega)} \le C(\|F\|_{L^2(\Omega)} + \|\rho\|_{L^2(\Omega)})$ *with a constant* $C = C(\Omega, \tilde\Omega)$*.*

Proof. We set $\Sigma := \tilde\Omega \setminus \overline{\Omega}$, it has the two boundary parts $\partial\Omega$ and $\Gamma := \partial\tilde\Omega$.

We must construct $\tilde F$ on Σ. We use an extension operator; there exists a bounded linear extension operator $\mathcal{E}\colon H^1(\Sigma, \mathbb{R}) \to H^1(\tilde\Omega, \mathbb{R})$ such that $\hat\varphi := \mathcal{E}(\varphi)$ satisfies $\hat\varphi|_\Sigma = \varphi$. Such an operator can be constructed locally with charts, using the technique of "flattening the boundary" and reflecting the function in an even way to the other side.

We use the space $H^1_\Gamma(\Sigma)$ of H^1-functions on Σ that vanish on Γ. With the extension operator $\mathcal{E}$ we define $p \in H^1_\Gamma(\Sigma)$ as the solution of

$$\int_\Sigma \nabla p \cdot \nabla\varphi = -\int_\Omega F \cdot \nabla\mathcal{E}(\varphi) - \int_\Omega \rho\, \mathcal{E}(\varphi) \qquad \forall \varphi \in H^1_\Gamma(\Sigma)\,. \tag{4.5}$$

This problem can be solved with $p \in H^1_\Gamma(\Sigma)$ with the lemma of Lax–Milgram since the right-hand side defines a continuous linear form on $H^1_\Gamma(\Sigma)$. We set

$$\tilde F(x) := \begin{cases} F(x) & \text{for } x \in \Omega\,, \\ \nabla p(x) & \text{for } x \in \Sigma\,. \end{cases}$$

It remains to calculate the divergence of $\tilde F$ in $\tilde\Omega$. For an arbitrary test-function $\tilde\psi \in H^1_0(\tilde\Omega)$ and its restriction $\psi := \tilde\psi|_\Sigma \in H^1_\Gamma(\Sigma)$, we calculate

$$\begin{aligned} \int_{\tilde\Omega} \tilde F \cdot \nabla\tilde\psi &= \int_\Omega F \cdot \nabla\tilde\psi + \int_\Sigma \nabla p \cdot \nabla\psi \\ &\overset{(4.5)}{=} \int_\Omega F \cdot \nabla\tilde\psi - \int_\Omega F \cdot \nabla\mathcal{E}(\psi) - \int_\Omega \rho\, \mathcal{E}(\psi) \\ &= \int_\Omega F \cdot \nabla[\tilde\psi - \mathcal{E}(\psi)] - \int_\Omega \rho\, \mathcal{E}(\psi) \\ &= -\int_\Omega \rho\, [\tilde\psi - \mathcal{E}(\psi)] - \int_\Omega \rho\, \mathcal{E}(\psi) = -\int_\Omega \rho\, \tilde\psi\,. \end{aligned}$$

This verifies that the divergence of $\tilde F$ coincides with the trivial extension of ρ. ☐

Appendix A. Proof of compactness for vanishing tangential traces

The compactness proof is simpler when the condition $\nu \times u|_\Gamma = 0$ is imposed.

Proposition A.1 (Maxwell compactness theorem for vanishing tangential trace and $\varepsilon = 1$)**.** *Let* $\Omega \subset \mathbb{R}^3$ *be a bounded Lipschitz domain. Then, the following embedding is compact:*

$$H_0(\mathrm{curl}, \Omega) \cap H(\mathrm{div}, \Omega) \xrightarrow{cpt.} L^2(\Omega, \mathbb{C}^3)\,. \tag{A.1}$$

Proof. We consider a bounded sequence u_j in the space on the left and recall that this implies that u_j, $\mathrm{curl}\, u_j$, $\mathrm{div}\, u_j$ are bounded sequences in $L^2(\Omega)$ and that tangential boundary values of u_j vanish.

The set Ω is bounded, we can therefore choose a radius $R > 0$ such that Ω is compactly contained in the open ball with this radius, $\overline{\Omega} \subset B_R := B_R(0) \subset \mathbb{R}^3$.

We extend each function u_j trivially to B_R and denote this extension by $\tilde u_j$. Since the tangential boundary values of u_j vanish, there holds $\tilde u_j \in H(\mathrm{curl}, \mathbb{R}^3)$. With the trivial extension $\tilde f_j$ of f_j, there holds $\mathrm{curl}\, \tilde u_j = \tilde f_j$.

Step 1: Modification with a gradient. The extension $\tilde{u}_j$ may have jumps in its normal component across $\Gamma := \partial\Omega$. Nonetheless, by $L^2(B_R)$-boundedness of $\tilde{u}_j$, the functional $\operatorname{div} \tilde{u}_j$ is of class $H^{-1}(B_R)$ and we can solve the problem

$$\Delta\tilde{\phi}_j = \operatorname{div} \tilde{u}_j \qquad \text{in } H^{-1}(B_R)$$

with some scalar function $\tilde{\phi} \in H_0^1(B_R)$. We consider $\tilde{w}_j := \tilde{u}_j - \nabla\tilde{\phi}_j$. By the choice of $\tilde{\phi}$, there holds $\operatorname{div}(\tilde{w}_j) = 0$ on B_R. Since $\tilde{u}_j$ is bounded in $L^2(B_R)$, the solution sequence $\tilde{\phi}_j$ is bounded in $H^1(B_R)$. We have therefore constructed a bounded sequence $\tilde{w}_j \in L^2(B_R, \mathbb{C}^3)$ with vanishing divergence and with the rotation $\operatorname{curl} \tilde{w}_j = \operatorname{curl} \tilde{u}_j = \tilde{f}_j$.

Because of $\Delta = -\operatorname{curl}\operatorname{curl} + \nabla \operatorname{div}$, the function $\tilde{w}_j \in L^2(B_R, \mathbb{C}^3)$ satisfies, in the sense of distributions, $\Delta\tilde{w}_j = -\operatorname{curl} \tilde{f}_j \in H^{-1}(B_R)$. The sequence $\tilde{w}_j$ is therefore locally of class H^1 in B_R (Caccioppoli's inequality). This implies that $w_j := \tilde{w}_j|_\Omega$ is bounded in $H^1(\Omega)$. Upon passing to a subsequence, we can assume the strong convergence of the sequence w_j in $L^2(B_R, \mathbb{C}^3)$. Since w_j converges strongly, the lemma is proven when we show for $\phi_j := \tilde{\phi}_j|_\Omega$ that $\nabla\phi_j$ converges strongly in $L^2(\Omega)$, possibly along a further subsequence.

Step 2: Regularity along $\partial\Omega$. Let us analyze the regularity properties of ϕ_j along $\Gamma = \partial\Omega$. We have bounds for $\nu \times w_j|_\Gamma \in L^2(\partial\Omega)$ because of the boundedness of $w_j \in H^1(\Omega)$ and classical trace theorems. Because of $\nu \times u_j|_\Gamma = 0$, the sequence $\nu \times \nabla\phi_j|_\Gamma = -\nu \times w_j|_\Gamma \in L^2(\partial\Omega)$ is bounded. This provides a bound for $\phi_j|_\Gamma \in H^1(\Gamma)$.

It remains to analyze the Dirichlet problem

$$\Delta\phi = g \quad \text{ in } \Omega\,, \qquad \phi = \psi \text{ on } \Gamma\,.$$

By classical elliptic theory, the solution operator to this problem is a bounded linear operator $\mathcal{T}\colon H^{-1}(\Omega) \times H^{1/2}(\Gamma) \ni (g, \psi) \mapsto \phi \in H^1(\Omega)$. We have the compact embeddings $L^2(\Omega) \xrightarrow{\text{cpt.}} H^{-1}(\Omega)$ and $H^1(\Gamma) \xrightarrow{\text{cpt.}} H^{1/2}(\Gamma)$. They imply that the operator $\mathcal{T}$ on these better function spaces is compact,

$$\tilde{\mathcal{T}}\colon L^2(\Omega) \times H^1(\Gamma) \ni (g, \psi) \mapsto \phi \in H^1(\Omega) \quad \text{compact}\,.$$

As a solution of the Dirichlet problem, $\phi_j = \tilde{\mathcal{T}}(g_j, \phi_j|_\Gamma)$, the sequence ϕ_j has a subsequence that converges in $H^1(\Omega)$. □

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