Lecture notes:
Existence result for Maxwell's equations on bounded Lipschitz domains

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Abstract: We give a proof for the existence of weak solutions to time-harmonic Maxwell's equations on bounded Lipschitz domains. This result is well-known; our aim here is to present, in a concise way, the arguments that lead from a compactness result for divergence free functions and a Helmholtz decomposition to the existence result.

1. Introduction

In this note, we study time-harmonic Maxwell's equations. Given is a bounded domain $\Omega \subset \mathbb{R}^3$, two coefficient functions $\varepsilon \in L^\infty(\Omega, \mathbb{R})$ and $\mu \in L^\infty(\Omega, \mathbb{R})$, a frequency $\omega > 0$ and right-hand sides $f_h, f_e \colon \Omega \to \mathbb{C}^3$. We seek for functions $E, H \colon \Omega \to \mathbb{C}^3$ that satisfy

$$\operatorname{curl} E = i\omega\mu H + f_h \quad \text{in } \Omega\,, \tag{1.1a}$$

$$\operatorname{curl} H = -i\omega\varepsilon E + f_e \quad \text{in } \Omega\,, \tag{1.1b}$$

with a homogeneous tangential boundary condition for E,

$$E \times \nu = 0 \quad \text{on } \partial\Omega\,, \tag{1.1c}$$

where ν is the exterior normal vector on $\partial\Omega$. The boundary condition (1.1c) is understood as $E \in H_0(\operatorname{curl}, \Omega)$. The weak solution concept and the spaces $H_0(\operatorname{curl}, \Omega)$ and $H(\operatorname{curl}, \Omega)$ are defined below in (1.2)–(1.4).

1.1. Main result.

We present and prove a result on the existence of solutions to the Maxwell system (1.1). We will actually provide two different proofs, one with Fredholm operator theory, one with a limiting absorption principle. We emphasize that the result is classical, [3] is one of the more recent references. Our aim here is to present a short derivation.

Theorem 1.1 (Existence and uniqueness). *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and $\omega > 0$ be a frequency. Let $\varepsilon \in L^\infty(\Omega, \mathbb{R})$ and $\mu \in L^\infty(\Omega, \mathbb{R})$ be coefficient functions such that, for some $c_0 > 0$, there holds $\varepsilon \geq c_0$ and $\mu \geq c_0$. We assume that system (1.1) with $f_h = 0$ and $f_e = 0$ has only the trivial solution.*

Then, for every $f_h, f_e \in L^2(\Omega, \mathbb{C}^3)$, system (1.1) has a unique weak solution $(E, H) \in H_0(\operatorname{curl}, \Omega) \times H(\operatorname{curl}, \Omega)$.

We note that the uniqueness follows immediately from the linearity of the Maxwell system and the assumption of uniqueness for $f_h = f_e = 0$.

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In order to keep this exposition as simple as possible, we assume $\operatorname{div}(f_h) = 0$ in Ω and $f_h \cdot \nu = 0$ on $\partial\Omega$. This is understood in the weak sense, $\int_\Omega f_h \cdot \nabla\psi = 0$ for every $\psi \in H^1(\Omega)$. With the help of the Helmholtz decomposition, it can be shown that this assumption is not a loss of generality.

1.2. **Weak form.** To formulate (1.1) in a weak sense, we use the space of L^2-functions such that the distributional curl of the function is again of class L^2. More precisely, we use

$$H(\operatorname{curl},\Omega) := \left\{u \in L^2(\Omega,\mathbb{C}^3) \mid \operatorname{curl} u \in L^2(\Omega,\mathbb{C}^3)\right\} . \tag{1.2}$$

This is a Hilbert space with the norm $\|u\|^2_{H(\operatorname{curl},\Omega)} := \int_\Omega \{|u|^2 + |\operatorname{curl} u|^2\}$ and the scalar product $\langle u,\varphi\rangle := \langle u,\varphi\rangle_{L^2(\Omega)} + \langle \operatorname{curl} u, \operatorname{curl}\varphi\rangle_{L^2(\Omega)}$.

Let us motivate a weak formulation of the Maxwell system. We choose $\phi \in H(\operatorname{curl},\Omega)$ arbitrarily and use $\varepsilon^{-1}\operatorname{curl}\phi$ as a test-function for (1.1b). On the right-hand side occurs an integral over $E\cdot\operatorname{curl}\phi$, which is identical to the integral over $\operatorname{curl} E\cdot\phi$ because of the boundary condition (1.1c). Replacing $\operatorname{curl} E$ with the expression of (1.1a), we find

$$\int_\Omega \left\{\varepsilon^{-1}\operatorname{curl} H\cdot\operatorname{curl}\phi - \omega^2\mu H\cdot\phi\right\} = \int_\Omega \left\{-i\omega f_h\cdot\phi + \varepsilon^{-1} f_e\cdot\operatorname{curl}\phi\right\} \quad \forall\, \phi\,, \tag{1.3}$$

where the test-functions are all $\phi \in H(\operatorname{curl},\Omega)$. We choose (1.3) as the weak form of system (1.1).

Note that, once a solution H of (1.3) is found, E can be defined with formula (1.1b) and the pair (E,H) solves the coupled system. The tangential boundary condition $E\times\nu = 0$ on $\partial\Omega$ is satisfied in the form $E \in H_0(\operatorname{curl},\Omega)$, where

$$H_0(\operatorname{curl},\Omega) := \left\{u \in H(\operatorname{curl},\Omega) \,\middle|\, \int_\Omega \operatorname{curl}\, u\cdot\eta = \int_\Omega u\cdot\operatorname{curl}\,\eta \;\; \forall \eta \in H(\operatorname{curl},\Omega)\right\} . \tag{1.4}$$

Our aim is to prove Theorem 1.1 with the help of the compactness result of Lemma 1.2 and with the Helmholtz decomposition of Lemma 1.3.

1.3. **Tool I: Compactness.** The following lemma is classical in the theory of spaces of functions with curl in L^2.

Lemma 1.2 (Compactness). *Let Ω be a bounded Lipschitz domain and let $\mu \in L^\infty(\Omega)$ be bounded below by some positive constant. Then the space*

$$Y := \left\{u \in H(\operatorname{curl},\Omega) \,\middle|\, \int_\Omega \mu\, u\cdot\nabla\psi = 0 \text{ for all } \psi \in H^1(\Omega)\right\} \tag{1.5}$$

is compactly imbedded in $L^2(\Omega,\mathbb{C}^3)$.

Only standard methods are necessary to prove Theorem 1.1. In order to make that clear, we give a proof in Section 4. Proposition 4.1 in Section 4 is even more general in the sense that only the L^2-control of the divergence of μu is demanded (and not that the divergence vanishes).

Comments on the literature regarding Lemma 1.2. The lemma is stated in [4, Lemma A.1] with our assumption on the coefficient. A very similar version is [3, Corollary 4.36], where positive definite symmetric matrix valued coefficients $\mu \in L^\infty(\Omega,\mathbb{C}^{3\times3})$ are allowed. We remark that closely related results are also stated and proved for the subset of $H_0(\operatorname{curl},\Omega)$ in [3, Theorem 4.24] and [5, Theorem 4.7], the latter with additional regularity assumptions on the coefficient.

1.4. **Tool II: Helmholtz decomposition.** The space Y of (1.5) is complemented with the space of gradients,

$$G := \left\{ u \in H(\mathrm{curl}, \Omega) \,\middle|\, \exists \psi \in H^1(\Omega) \colon u = \nabla\psi \right\} . \tag{1.6}$$

The choice is such that Y is orthogonal to G in the space $L^2(\Omega, \mathbb{C}^3)$ with the weighted scalar product $\langle u, v\rangle = \int_\Omega \mu\, u \cdot \overline{v}$. Furthermore, since the (distributional) curl of a gradient always vanishes, $\mathrm{curl}(\nabla\psi) = 0$, the two subspaces are also orthogonal in $X := H(\mathrm{curl}, \Omega)$ with the scalar product $\langle u, v\rangle_X := \int_\Omega \{\mu\, u \cdot \overline{v} + \mathrm{curl}\, u \cdot \mathrm{curl}\, \overline{v}\}$. By construction, Y is the $\langle ., . \rangle_X$-orthogonal complement of G, we may write this as $Y = G^\perp$. We therefore have the following result:

Lemma 1.3 (Helmholtz decomposition)**.** *The space $X := H(\mathrm{curl}, \Omega)$ has the orthogonal decomposition $X = Y \oplus G$. In particular, an arbitrary element $u \in X$ can be written uniquely as $u = v + \nabla\psi$ with $v \in Y$ and $\psi \in H^1(\Omega)$.*

2. Existence proof with Fredholm operator theory

In our first existence proof, we re-formulate problem (1.3) with the Helmholtz decomposition. We seek solutions H of (1.3) in the space Y of (1.5). We recall that the condition $H \in Y$ encodes that the divergence of μH vanishes and that $H \cdot \nu$ vanishes along the boundary.

Lemma 2.1 (Equivalent formulation in Y)**.** *The Maxwell system in the form (1.3) is equivalent to: Find $H \in Y$ with*

$$\int_\Omega \left\{\varepsilon^{-1}\, \mathrm{curl}\, H \cdot \mathrm{curl}\, \varphi - \omega^2 \mu H \cdot \varphi\right\} = \int_\Omega \left\{-i\omega f_h \cdot \varphi + \varepsilon^{-1} f_e \cdot \mathrm{curl}\, \varphi\right\} \quad \forall\, \varphi \in Y\, . \tag{2.1}$$

Proof. Let H be a solution of (1.3). For arbitrary $\psi \in H^1(\Omega, \mathbb{R})$, we define $\varphi = \nabla\psi$; because of the distributional equality $\mathrm{curl}\, \nabla\psi = 0$, there holds $\varphi \in H(\mathrm{curl}, \Omega)$. Using φ as a test-function in (1.3), we note that all terms except for $\int_\Omega \omega^2 \mu H \cdot \varphi$ vanish because of $\mathrm{curl}\, \varphi = 0$ and $\int_\Omega f_h \cdot \nabla\psi = 0$ for $\psi \in H^1(\Omega)$. The result is that μH is orthogonal to gradients, i.e., $H \in Y$. Since the space of test-functions is smaller in (2.1) than in (1.3), this implies that (2.1) holds.

Vice versa, let H be a solution of (2.1). Given an arbitrary test-function $\phi \in X = H(\mathrm{curl}, \Omega)$, we write $\phi = \varphi + \nabla\psi$ with $\varphi \in Y$ and $\psi \in H^1(\Omega)$, see Lemma 1.3. We use that (1.3) is linear in ϕ, we can treat all contributions separately. Inserting $\nabla\psi$ in (1.3), arguing as above and exploiting $H \in Y$, we find that all terms vanish. Inserting φ, the equality holds because of (2.1). This shows that (1.3) is satisfied for the test-function ϕ. Since ϕ was arbitrary, H is a solution of (1.3). □

The re-formulation (2.1) is useful since Lemma 1.2 provides compactness of Y.

Proof of Theorem 1.1 with Fredholm operators. On Y, we define sesquilinear forms $a, b \colon Y \times Y \to \mathbb{C}$ and a linear right-hand side $f \colon Y \to \mathbb{C}$ by setting, for every $u, \phi \in Y$,

$$a(u, \phi) := \int_\Omega \left\{u \cdot \overline{\phi} + \varepsilon^{-1}\, \mathrm{curl}\, u \cdot \mathrm{curl}\, \overline{\phi}\right\} ,$$

$$b(u, \phi) := \int_\Omega \left\{u \cdot \overline{\phi} + \omega^2 \mu\, u \cdot \overline{\phi}\right\} , \quad f(\phi) := \int_\Omega \left\{-i\omega f_h \cdot \overline{\phi} + \varepsilon^{-1} f_e \cdot \mathrm{curl}\, \overline{\phi}\right\} .$$

Lemma 2.1 yields: The Maxwell system (1.3) is equivalent to: Find $H \in Y$ with

$$a(H,\varphi) - b(H,\varphi) = f(\varphi) \qquad \forall\, \varphi \in Y\,. \tag{2.2}$$

This is the equation that we have to solve.

The sesquilinear form a defines a map $A\colon Y \to Y'$ from Y into the dual space Y' by $Au := a(u,\cdot)$. By definition of the scalar product in $Y \subset X = H(\operatorname{curl},\Omega)$, the form a is coercive on Y. The Lemma of Lax–Milgram implies that $A\colon Y \to Y'$ is invertible.

We now exploit that the embedding $\iota\colon Y \to L^2(\Omega,\mathbb{C}^3)$ is compact, see Lemma 1.2. The multiplication map $B\colon u \mapsto (1+\omega^2\mu)\, u$ corresponding to b is linear and bounded as a map $B\colon L^2(\Omega,\mathbb{C}^3) \to L^2(\Omega,\mathbb{C}^3)$. We denote the concatenation with an embedding into Y' with the same letter, $B\colon L^2(\Omega,\mathbb{C}^3) \to Y'$.

The field $H \in Y$ solves (2.2) if and only if

$$A\,H - (B\circ\iota)\,H = f \qquad \text{in } Y'\,.$$

Applying A^{-1}, we find the equivalent relation

$$(\operatorname{id} - A^{-1}\circ B\circ\iota)H = A^{-1}f \qquad \text{in } Y\,.$$

The operator $A^{-1}\circ B\circ\iota$ is compact, since ι is compact and the other operators are continuous. This implies that the operator $F := \operatorname{id} - A^{-1}\circ B\circ\iota$ is a Fredholm operator of index zero, see [1, Theorem 11.8].

One assumption in Theorem 1.1 is that the homogeneous system has only the trivial solution. Since all our re-formulations of the Maxwell system are equivalent, this implies that $F = \operatorname{id} - A^{-1}\circ B\circ\iota$ is injective. Since the index is zero, the injectivity implies the surjectivity. Surjectivity of F implies that for every right-hand side there exists a solution of (2.2). Since the formulations are equivalent, we have also found a solution of (1.1) and Theorem 1.1 is proved □

3. Existence proof with a limiting absorption principle

In this section, we prove Theorem 1.1 with a limiting absorption principle. A limiting absorption principle considers the equation with a damping term, which is sent to zero. In this text, we use a small constant $\delta > 0$ and replace the pre-factor $\omega^2\mu$ in equation (1.3) by $\omega^2\mu - i\delta$. This equation has a solution H_δ. We find a limit $H = \lim_{\delta\to 0} H_\delta$, which is a solution of (1.3).

In the first step, we seek a solution $H_\delta \in H(\operatorname{curl},\Omega)$ of

$$\int_\Omega \left\{\varepsilon^{-1}\operatorname{curl} H_\delta \cdot \operatorname{curl}\phi - (\omega^2\mu - i\delta)H_\delta\cdot\phi\right\} = \int_\Omega \left\{-i\omega f_h\cdot\phi + \varepsilon^{-1} f_e\cdot\operatorname{curl}\phi\right\} \quad \forall\,\phi\,, \tag{3.1}$$

where the test-functions are arbitrary functions $\phi \in H(\operatorname{curl},\Omega)$.

Proof of Theorem 1.1 with limiting absorption. Lemma 3.1 provides a unique solution H_δ of (3.1). Lemma 3.2 shows that the sequence H_δ is bounded in $H(\operatorname{curl},\Omega)$ (in the setting of Theorem 1.1, where it is assumed that (1.3) has only the trivial solution for $f_h = f_e = 0$). Since $H(\operatorname{curl},\Omega)$ is reflexive, there exists a subsequence of H_δ, which converges weakly to some limit $H \in H(\operatorname{curl},\Omega)$. The weak convergence allows us to perform the limit $\delta \to 0$ in (3.1) along the subsequence. We obtain that H solves (1.3). □

Lemma 3.1 (Existence of a solution for the problem with absorption)**.** *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and $\omega \in \mathbb{R} \setminus \{0\}$ a frequency. Let $\varepsilon \in L^\infty(\Omega, \mathbb{R})$ and $\mu \in L^\infty(\Omega, \mathbb{R})$ be coefficient functions such that, for some $c_0 > 0$, there holds $\varepsilon \geq c_0$ and $\mu \geq c_0$. Then there exists $\delta_0 > 0$ such that, for every $\delta \in (0, \delta_0)$, equation (3.1) has a unique weak solution $H_\delta \in H(\operatorname{curl}, \Omega)$.*

Proof. We define the bilinear form a_δ on $H(\operatorname{curl}, \Omega)$ by setting, for $u, \varphi \in H(\operatorname{curl}, \Omega)$,

$$a_\delta(u, \varphi) := \int_\Omega \left\{ \varepsilon^{-1} \operatorname{curl} u \cdot \operatorname{curl} \overline{\varphi} + (i\delta - \omega^2 \mu) u \cdot \overline{\varphi} \right\} .$$

The form a_δ allows us to rewrite (3.1) as

$$a_\delta(H_\delta, \phi) = \int_\Omega \left\{ -i\omega f_h \cdot \overline{\phi} + \varepsilon^{-1} f_e \cdot \operatorname{curl} \overline{\phi} \right\} \qquad \forall\, \phi \in H(\operatorname{curl}, \Omega) . \tag{3.2}$$

We calculate, for arbitrary $u \in H(\operatorname{curl}, \Omega)$,

$$\begin{aligned} &\operatorname{Im} a_\delta(u, u) = \delta \|u\|^2_{L^2(\Omega)} , \\ &\operatorname{Re} a_\delta(u, u) \geq \operatorname{ess\,sup}(\varepsilon)^{-1} \| \operatorname{curl} u\|^2_{L^2(\Omega)} - \omega^2 \operatorname{ess\,sup}(\mu) \|u\|^2_{L^2(\Omega)} . \end{aligned}$$

We observe the following fact in $\mathbb{R}^2 \equiv \mathbb{C}$: For every vector $x = (x_1, x_2) \in \mathbb{R}^2$ and every $s \in [0, 1]$, there holds $|x| \geq \max\{|x_1|, |x_2|\} \geq (1 - s)|x_1| + s|x_2|$. This inequality allows to calculate, with $s = \delta^2$,

$$\begin{aligned} |a_\delta(u, u)| &\geq (1 - \delta^2) \operatorname{Im} a_\delta(u, u) + \delta^2 \operatorname{Re} a_\delta(u, u) \\ &\geq (1 - \delta^2) \delta \|u\|^2_{L^2(\Omega)} + \delta^2 \left(\operatorname{ess\,sup}(\varepsilon)^{-1} \| \operatorname{curl} u\|^2_{L^2(\Omega)} - \omega^2 \operatorname{ess\,sup}(\mu) \|u\|^2_{L^2(\Omega)} \right) . \end{aligned}$$

Choosing $\delta_0 > 0$ small, we achieve $(1 - \delta^2)\delta \geq 2\delta^2 \omega^2 \operatorname{ess\,sup}(\mu)$ for all $\delta < \delta_0$, and obtain that a_δ is coercive. Problem (3.2) for H_δ can therefore be solved with the Lemma of Lax–Milgram. □

Lemma 3.2 (Boundedness of solutions to the problem with absorption)**.** *Let the assumptions of Lemma 3.1 be satisfied. For a sequence $\delta \to 0$, let $H_\delta \in H(\operatorname{curl}, \Omega)$ be the corresponding sequence of solutions of (3.1). Suppose that (1.3) for $f_h = f_e = 0$ has only the trivial solution $H = 0$. Then, H_δ is bounded in $H(\operatorname{curl}, \Omega)$.*

Proof. Step 1: Preparation. For a contradiction argument, we assume that there exists a subsequence H_δ such that $\|H_\delta\|_{H(\operatorname{curl},\Omega)} \to \infty$. We use H_δ as a test-function in (3.2) and use the upper bound $C_\varepsilon := \operatorname{ess\,sup}(\varepsilon)$ to obtain

$$\begin{aligned} C_\varepsilon^{-1} \| \operatorname{curl} H_\delta\|^2_{L^2(\Omega)} &\leq \int_\Omega \varepsilon^{-1} \operatorname{curl} H_\delta \cdot \operatorname{curl} \overline{H}_\delta \\ &= \int_\Omega -(i\delta - \omega^2 \mu) H_\delta \cdot \overline{H_\delta} + \int_\Omega \left\{ -i\omega f_h \cdot \overline{H_\delta} + \varepsilon^{-1} f_e \cdot \operatorname{curl} \overline{H_\delta} \right\} . \end{aligned}$$

For arbitrary $\lambda > 0$, we continue this calculation with Young's inequality and find

$$C_\varepsilon^{-1} \| \operatorname{curl} H_\delta\|^2_{L^2(\Omega)} \leq C \|H_\delta\|^2_{L^2(\Omega)} + C_\lambda + \lambda \| \operatorname{curl} H_\delta\|^2_{L^2(\Omega)} ,$$

for some C depending on ω, f_h and μ, and C_λ depending on ε, f_e and λ. Choosing $\lambda = C_\varepsilon^{-1}/2$ and subtracting the term $\lambda \| \operatorname{curl} H_\delta\|^2_{L^2(\Omega)}$ on both sides, we find, for some constant C, the inequality $\| \operatorname{curl} H_\delta\|^2_{L^2(\Omega)} \leq C\,(1 + \|H_\delta\|^2_{L^2(\Omega)})$.

In particular: The divergence $\|H_\delta\|_{H(\operatorname{curl},\Omega)} \to \infty$ implies $\|H_\delta\|_{L^2(\Omega)} \to \infty$.

Step 2: Normalization. We normalize the sequence H_δ and consider the new sequence $\tilde H_\delta := H_\delta / \|H_\delta\|_{L^2(\Omega,\mathbb{C}^3)}$. This sequence satisfies

$$a_\delta(\tilde H_\delta, \phi) = \|H_\delta\|^{-1}_{L^2(\Omega,\mathbb{C}^3)} \int_\Omega \left\{ -i\omega f_h \cdot \overline{\phi} + \varepsilon^{-1} f_e \cdot \operatorname{curl} \overline{\phi} \right\} \quad \forall\, \phi \in H(\operatorname{curl}, \Omega)\,. \tag{3.3}$$

The result of Step 1 together with $\|\tilde H_\delta\|_{L^2(\Omega,\mathbb{C}^3)} = 1$ shows that $\|\operatorname{curl} \tilde H_\delta\|^2_{L^2(\Omega)}$ is bounded.

Since $H(\operatorname{curl}, \Omega)$ is reflexive, there exists a subsequence of $\tilde H_\delta$, which converges weakly to some limit $\tilde H \in H(\operatorname{curl}, \Omega)$, in particular $\tilde H_\delta \rightharpoonup \tilde H$ in $L^2(\Omega)$ and $\operatorname{curl} \tilde H_\delta \rightharpoonup \operatorname{curl} \tilde H$ in $L^2(\Omega)$. The weak convergence allows us to perform the limit $\delta \to 0$ in (3.3) along the subsequence. We obtain that the limit $\tilde H$ solves (1.3) for $f_h = f_e = 0$. Our assumption was that there is no non-trivial solution to the homogeneous problem; this implies $\tilde H = 0$.

Step 3: Strong convergence. In this step we show the strong convergence of $\tilde H_\delta$ in $L^2(\Omega, \mathbb{C}^3)$ along a subsequence. Once this is obtained, we have the desired contradiction: $\|\tilde H_\delta\|_{L^2(\Omega)} = 1$ is in conflict with the strong convergence $\tilde H_\delta \to \tilde H = 0$.

In order to show the strong convergence, we decompose $\tilde H_\delta$ by means of the Helmholtz decomposition of Lemma 1.3: $\tilde H_\delta = \tilde H^Y_\delta + \nabla \psi_\delta$ for $\tilde H^Y_\delta \in Y$ and $\psi_\delta \in H^1(\Omega)$. The compactness of Y shown in Lemma 1.2 allows us to pass to a subsequence such that $\tilde H^Y_\delta$ converges strongly in $L^2(\Omega, \mathbb{C}^3)$.

Let us use the test-function $\phi = \nabla \overline{\psi_\delta}$ in (3.1). Since the curl of a gradient vanishes and since we assumed that f_h is orthogonal to gradients, we obtain

$$\int_\Omega (\mu - i\omega^{-2}\delta) \tilde H_\delta \cdot \nabla \overline{\psi_\delta} = 0\,.$$

Inserting the decomposition $\tilde H_\delta = \tilde H^Y_\delta + \nabla \psi_\delta$, we find

$$i \int_\Omega \omega^{-2} \delta \tilde H_\delta \cdot \nabla \overline{\psi_\delta} - \int_\Omega \mu |\nabla \psi_\delta|^2 = \int_\Omega \mu \tilde H^Y_\delta \cdot \nabla \overline{\psi_\delta} = 0\,, \tag{3.4}$$

where we used the property $\tilde H^Y_\delta \in Y$ in the last equality. The lower bound $\mu \geq c_0 > 0$ allows us to obtain from (3.4)

$$c_0 \|\nabla \psi_\delta\|^2_{L^2(\Omega)} \leq \int_\Omega \mu |\nabla \psi_\delta|^2 = i \int_\Omega \omega^{-2} \delta \tilde H_\delta \cdot \nabla \overline{\psi_\delta}\,.$$

The Cauchy-Schwarz inequality and the normalization $\|\tilde H_\delta\|_{L^2(\Omega)} = 1$ imply

$$c_0 \|\nabla \psi_\delta\|^2_{L^2(\Omega)} \leq \delta \omega^{-2} \|\tilde H_\delta\|_{L^2(\Omega)} \|\nabla \psi_\delta\|_{L^2(\Omega)} = \delta \omega^{-2} \|\nabla \psi_\delta\|_{L^2(\Omega)}\,.$$

Dividing by $\|\nabla \psi_\delta\|_{L^2(\Omega)}$ if this term is different from zero, we find $\|\nabla \psi_\delta\|_{L^2(\Omega)} \to 0$ as $\delta \to 0$. This is the desired strong convergence of $\nabla \psi_\delta$.

The strong convergence of $\tilde H^Y_\delta$ together with the strong convergence of $\nabla \psi_\delta$ implies the strong convergence of $\tilde H_\delta = \tilde H^Y_\delta + \nabla \psi_\delta$ in $L^2(\Omega)$. This provides the desired contradiction and concludes the proof. □

4. Compactness property

We include a proof for the compactness result in order to have this exposition self-contained. The compactness has some relations with the div-curl lemma, sometimes called compensated compactness. Knowledge on the curl and on the divergence of a function somehow controls all derivatives. Loosely speaking, this property is already suggested by the relation $\Delta = -\operatorname{curl}\operatorname{curl} + \nabla \operatorname{div}$.

Proposition 4.1 (Compactness when rotation and divergence are bounded in L^2)**.** *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain, let $u_j \in L^2(\Omega, \mathbb{C}^3)$ be a sequence such that the distributional curl and the distributional divergence are bounded in L^2,*

$$\operatorname{curl} u_j =: f_j \in L^2(\Omega, \mathbb{C}^3) \text{ bounded,} \tag{4.1}$$

$$\operatorname{div} u_j =: g_j \in L^2(\Omega, \mathbb{C}) \text{ bounded.} \tag{4.2}$$

Furthermore, we demand vanishing normal boundary values of u_j by imposing $\int_\Omega u_j \cdot \nabla\varphi = -\int_\Omega g_j\, \varphi$ for all $\varphi \in H^1(\Omega, \mathbb{C})$. Then, there exists a subsequence $j \to \infty$ and a limit function $u \in L^2(\Omega, \mathbb{C}^3)$ such that $u_j \to u$ in $L^2(\Omega, \mathbb{C}^3)$ along this subsequence.

Proof. We first prove the Proposition in Steps 1 to 3 under the assumption that Ω is simply connected. The assumption of simple connectedness is removed in Step 4. The bounded set Ω is contained in a large ball. For a radius $R > 0$ we start our proof with $\overline{\Omega} \subset \tilde{\Omega} := B_R(0) \subset \mathbb{R}^3$. Later on, we will choose a smaller open bounded set $\tilde{\Omega} \subset \mathbb{R}^3$.

Step 1: A vector potential for f_j. The divergence of u_j is trivially extended, we define $\tilde{g}_j \in L^2(\tilde{\Omega})$ as $\tilde{g}_j|_\Omega = g_j$ and $\tilde{g}_j|_\Omega = 0$ in $\tilde{\Omega} \setminus \Omega$. The extension $\tilde{f}_j$ of f_j (with vanishing divergence in Ω) is chosen such that $\operatorname{div} \tilde{f}_j$ vanishes in $\tilde{\Omega}$; we use here Lemma 4.2 and choose a smaller set $\tilde{\Omega}$ containing $\overline{\Omega}$, if necessary.

As a function with vanishing divergence, $\tilde{f}_j$ can be written with a vector potential $\tilde{w}_j \in L^2(\tilde{\Omega}, \mathbb{C}^3)$ on the larger domain $\tilde{\Omega}$,

$$\tilde{f}_j = \operatorname{curl} \tilde{w}_j\,. \tag{4.3}$$

The existence of such a vector potential is classical theory, $\tilde{w}_j$ can be constructed with the help of path integrals over the components of $\tilde{f}_j$, see [2] and [6] for modern references.

Step 2: Modification of the vector potential $\tilde{w}_j$. We want to modify the vector potential such that it is divergence-free. We find such a modified potential in the form $\tilde{W}_j = \tilde{w}_j - \nabla\tilde{\psi}_j$ where $\tilde{\psi}_j$ solves

$$\Delta\tilde{\psi}_j = \operatorname{div} \tilde{w}_j$$

in $\tilde{\Omega}$ (with homogeneous Dirichlet conditions on $\partial\tilde{\Omega}$) so that $\operatorname{div}(\tilde{W}_j) = 0$. Since $\tilde{w}_j$ is bounded in $L^2(\tilde{\Omega})$, we find $\tilde{\psi}_j$ bounded in $H^1(\tilde{\Omega})$. The result is a bounded sequence $\tilde{W}_j \in L^2(\tilde{\Omega}, \mathbb{C}^3)$ with the (distributional) rotation $\tilde{f}_j$ and vanishing divergence. We note that, because of $\Delta = -\operatorname{curl}\operatorname{curl} + \nabla\operatorname{div}$, the function $\tilde{W}_j \in H^1(\tilde{\Omega}, \mathbb{C}^3)$ satisfies, in the sense of distributions, $\Delta\tilde{W}_j = -\operatorname{curl} \tilde{f}_j \in H^{-1}(\tilde{\Omega})$. The sequence $\tilde{W}_j$ is therefore locally of class H^1. By possibly choosing a smaller open set $\tilde{\Omega}$ (still containing $\overline{\Omega}$), we achieve that the sequence $\tilde{W}_j \in H^1(\tilde{\Omega})$ is bounded (Caccioppoli's inequality). The restriction $W_j := \tilde{W}_j|_\Omega$ has the rotation f_j and vanishing divergence.

Since the sequence is bounded in $H^1(\tilde{\Omega}, \mathbb{C}^3)$, there exists a subsequence with strong convergence of $\tilde{W}_j$ in $L^2(\tilde{\Omega}, \mathbb{C}^3)$. We consider only this subsequence in the following.

Step 3: Construction of a scalar potential. We consider the function $v_j := u_j - W_j \in L^2(\Omega, \mathbb{R}^3)$ with vanishing rotation. Since Ω is simply connected, the function has a scalar potential ϕ_j,

$$v_j = \nabla\phi_j\,.$$

We have therefore written the original sequence as $u_j = W_j + \nabla\phi_j$. Since W_j converges strongly, the lemma is proven when we show that $\nabla\phi_j$ converges strongly in L^2 along a subsequence.

With the normal vector ν on $\partial\Omega$, the function ϕ_j is a solution of

$$\Delta\phi_j = g_j \text{ in } \Omega\,, \qquad \nu\cdot\nabla\phi_j = -\nu\cdot W_j \text{ on } \partial\Omega\,,$$

where we used $\nu\cdot u_j = 0$ on the boundary. This must be understood in the weak form: The potential ϕ_j is a solution of the problem

$$\int_\Omega \nabla\phi_j\cdot\nabla\varphi = -\int_\Omega g_j\,\varphi + \int_{\partial\Omega} W_j\,\varphi \tag{4.4}$$

for every $\varphi\in H^1(\Omega)$. To have uniqueness of solutions, we demand that the integral $\int_\Omega \phi_j = 0$ and define $H^1_*(\Omega)$ as the space of $H^1(\Omega)$-functions satisfying this condition. The integrability condition on g_j and W_j is that the right-hand side of (4.4) vanishes for constant functions φ. This condition is satisfied in our setting because of $\operatorname{div} W_j = 0$ and the assumption on u_j in the lemma.

The theorem of Lax–Milgram can be applied, the solution operator to this problem, $\mathcal{T}_0\colon (g_j, W_j)\mapsto\phi_j$, is a bounded linear operator $\mathcal{T}_0\colon (H^1(\Omega))'\times L^2(\partial\Omega)\to H^1_*(\Omega)$ (we suppress here that the argument must satisfy the integrability condition).

Additionally, we have the compactness of the embedding $L^2(\Omega)\subset(H^1(\Omega))'$ and the compactness of the trace operator $H^1(\Omega)\to L^2(\partial\Omega)$. This shows that the solution operator as a map $\mathcal{T}_1\colon L^2(\Omega)\times H^1(\Omega)\to H^1_*(\Omega)$ is compact. Since g_j and W_j are bounded in the left spaces, we conclude that $\phi_j = \mathcal{T}_1(g_j, W_j)$ has a convergent subsequence in $H^1(\Omega)$. This shows that $\nabla\phi_j$ has a convergent subsequence in $L^2(\Omega,\mathbb{R}^3)$ and concludes the proof for simply connected domains.

Step 4: Removing the assumption of simple connectedness. Let now Ω be an arbitrary bounded Lipschitz domain. We choose a finite family of simply connected Lipschitz subdomains $\Omega_k\subset\Omega$, $k = 1,\dots,K$ that cover Ω. We choose a subordinate family of smooth cut-off functions η_k. The lemma is then applied in each subdomain Ω_k to the sequence $u_j\eta_k$. □

The task in the subsequent lemma is to extend the function F to a function $\tilde F$ such that the normal component of $\tilde F\cdot\nu$ has no jump on $\partial\Omega$.

Lemma 4.2 (Extension with bounded divergence)**.** *Let Ω and $\tilde\Omega$ be Lipschitz domains in $\mathbb{R}^n$ with $\overline{\Omega}\subset\tilde\Omega$. Let $F\in L^2(\Omega,\mathbb{R}^n)$ be a function with a distributional divergence $\rho = \operatorname{div}(F)\in L^2(\Omega,\mathbb{R})$. Then, there exists an extension $\tilde F\in L^2(\tilde\Omega,\mathbb{R}^n)$ such that the distributional divergence $\tilde\rho = \operatorname{div}(\tilde F)$ satisfies $\tilde\rho\in L^2(\tilde\Omega,\mathbb{R})$. One can choose $\tilde F$ such that the function $\tilde\rho$ coincides with the trivial extension of ρ in a neighborhood of $\overline{\Omega}$.*

Proof. Since the problem can be localized, it suffices to consider the case that $\Sigma := \tilde\Omega\setminus\overline{\Omega}$ is a domain with the two boundary components $\Gamma := \partial\tilde\Omega$ and $\partial\Omega$.

We must construct $\tilde F$ on the exterior domain Σ. This proof uses an extension operator for functions $\varphi\in H^1(\Sigma,\mathbb{R})$. We use the fact that there exists a bounded extension operator $\mathcal{E}\colon H^1(\Sigma,\mathbb{R})\to H^1(\tilde\Omega,\mathbb{R})$ such that $\hat\varphi := \mathcal{E}(\varphi)$ satisfies $\hat\varphi|_\Sigma = \varphi$.

With the extension operator $\mathcal{E}$ we define $p\in H^1_\Gamma(\Sigma)$ as the solution of

$$\int_\Sigma \nabla p\cdot\nabla\varphi = -\int_\Omega F\cdot\nabla\mathcal{E}(\varphi) - \int_\Omega \rho\,\mathcal{E}(\varphi) \tag{4.5}$$

for all $\varphi \in H^1_\Gamma(\Sigma)$, where $H^1_\Gamma(\Sigma)$ is the space of H^1-functions on Σ that vanish on Γ. This problem can be solved with $p \in H^1_\Gamma(\Sigma)$ with the lemma of Lax–Milgram since the right-hand side defines a continuous linear form on $H^1_\Gamma(\Sigma)$. We set

$$\tilde{F}(x) := \begin{cases} F(x) & \text{for } x \in \Omega\,, \\ \nabla p(x) & \text{for } x \in \Sigma\,. \end{cases}$$

It remains to calculate the divergence of $\tilde{F}$ in $\tilde{\Omega}$. For an arbitrary test-function $\tilde{\psi} \in H^1(\tilde{\Omega})$ and its restriction $\psi := \tilde{\psi}|_\Sigma \in H^1(\Sigma)$ we calculate

$$\begin{aligned} \int_{\tilde{\Omega}} \tilde{F} \cdot \nabla \tilde{\psi} &= \int_\Omega F \cdot \nabla \tilde{\psi} + \int_\Sigma \nabla p \cdot \nabla \psi \\ &\overset{(4.5)}{=} \int_\Omega F \cdot \nabla \tilde{\psi} - \int_\Omega F \cdot \nabla \mathcal{E}(\psi) - \int_\Omega \rho\, \mathcal{E}(\psi) \\ &= \int_\Omega F \cdot \nabla [\tilde{\psi} - \mathcal{E}(\psi)] - \int_\Omega \rho\, \mathcal{E}(\psi) \\ &= -\int_\Omega \rho\, [\tilde{\psi} - \mathcal{E}(\psi)] - \int_\Omega \rho\, \mathcal{E}(\psi) = -\int_\Omega \rho\, \tilde{\psi}\,. \end{aligned}$$

This verifies that the divergence of $\tilde{F}$ coincides with the trivial extension of ρ. □

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