Lecture notes:
The Rellich Lemma on the Helmholtz equation

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Abstract: We present a proof of the Rellich Lemma in modern language, following the original proof of Franz Rellich from 1943. We modify the proof such that it is self-contained. In particular, it does not rely on the "well-known" properties of Bessel functions.

1 Introduction

For a spectral parameter $\lambda > 0$ and a radius $R > 0$, we are interested in solutions of

$$(\Delta + \lambda) u = 0 \tag{1.1}$$

in the domain $\Omega := \mathbb{R}^n \setminus B_R(0)$. The dimension $n \in \mathbb{N}$ is arbitrary with $n \geq 2$.

Even when we fix a boundary condition along $\partial B_R(0)$, we cannot expect that solutions are unique; we also have to impose some "boundary condition for $|x| \to \infty$", a radiation condition. On the other hand, when we include a condition at infinity (there are different choices), then uniqueness can hold. A possible statement of this kind is given by the following result.

Theorem 1.1 (Uniqueness for the Helmholtz equation in an exterior domain)**.** *Let u be a solution of the Helmholtz equation* (1.1) *in the domain $\Omega := \mathbb{R}^n \setminus B_R(0)$ with the boundary condition $u = 0$ on $\partial B_R(0)$ and the radiation condition*

$$\int_{\partial B_\rho} |u|^2 \to 0 \qquad \text{for } \rho \to \infty.$$

Then u vanishes identically.

Theorem 1.1 follows immediately from Lemma 2.1 below, which is the famous Rellich Lemma. We emphasize that the lemma can be used in very general situations, while Theorem 1.1 is formulated in a very special situation (imposing $u = 0$ on $\partial B_R(0)$) and serves just as an illustration how the Rellich Lemma can be used.

2 The Rellich Lemma

We essentially use the original formulation of Franz Rellich from 1943, formulated and proved in [1].

Lemma 2.1 (Rellich Lemma)**.** *For parameters $\lambda, R > 0$ and dimension $2 \leq n \in \mathbb{N}$, let $u : \Omega = \mathbb{R}^n \setminus B_R(0) \to \mathbb{R}$ be a solution of* (1.1)*. If u is a solution that is not vanishing identically, then there exists $r_1 > R$ and $c_R > 0$ such that*

$$\int_{B_\rho \setminus B_R} |u|^2 \geq c_R\, \rho \qquad \forall\ \rho \geq r_1\,. \tag{2.1}$$

Proof. Step 1: Simplification to a function v in one variable. Let $u : \Omega = \mathbb{R}^n \setminus B_R(0) \to \mathbb{R}$ be a nontrivial solution to (1.1). For a radius $r_0 > R$ we consider the restriction of u to the sphere $\{x \in \mathbb{R}^n \mid |x| = r_0\}$. By choosing r_0 appropriately, we may assume that u is not vanishing on this sphere. The spherical harmonics (basic properties are collected below in Section 3) form a basis of $L^2(S^{n-1})$. In particular, we find one particular spherical harmonic, which we call Y, with norm 1 in $L^2(S^{n-1})$, such that

$$\int_{|y|=1} u(r_0 y)\, Y(y)\, dS(y) \neq 0\,. \tag{2.2}$$

We define a function $v : [R, \infty) \to \mathbb{R}$ by setting

$$v(r) := \int_{|y|=1} u(ry)\, Y(y)\ dS(y)\,, \tag{2.3}$$

the choice of Y with (2.2) implies $v(r_0) \neq 0$. Lemma 2.1 is shown when v satisfies, for some $r_1 > r_0$ and $c_R > 0$,

$$\int_R^\rho v(r)^2\, r^{n-1}\, dr \geq c_R\, \rho \qquad \forall \rho > r_1\,. \tag{2.4}$$

Indeed, (2.4) implies (2.1): We calculate for arbitrary $\rho > r_1$, using polar coordinates in (i), $\|Y\| = 1$ and Cauchy-Schwarz in (ii),

$$\int_{B_\rho \setminus B_R} |u|^2 \overset{(i)}{=} \int_R^\rho \int_{|x|=1} |u(ry)|^2\, dS(y)\, r^{n-1}\, dr \overset{(ii)}{\geq} \int_R^\rho |v(r)|^2\, r^{n-1}\, dr \overset{(2.4)}{\geq} c_R\, \rho\,.$$

It is therefore sufficient to verify (2.4).

Step 2: An equation for v. We multiply equation (1.1) with the spherical harmonic Y and integrate over a sphere with radius $r > R$. We use the Laplace-Beltrami operator $\varDelta_S$ and the fact that the Laplace operator in spherical coordinates reads $\varDelta = r^{-(n-1)}\partial_r(r^{n-1}\partial_r) + r^{-2}\varDelta_S$; we sketch the proof in Section 3. We find

$$\begin{aligned} 0 &= \int_{\partial B_r(0)} (\varDelta + \lambda) u\ Y\, dS \\ &= \int_{\partial B_r(0)} r^{-(n-1)}\partial_r(r^{n-1}\partial_r u)\ Y + \int_{\partial B_r(0)} r^{-2}\varDelta_S u\ Y + \int_{\partial B_r(0)} \lambda u\ Y\,. \end{aligned}$$

One has to be careful how to read the term with the Laplace-Beltrami operator, we let $\varDelta_S$ only act on functions on the unit sphere. Using that $\varDelta_S$ is self-adjoint and that Y is an eigenfunction of $-\varDelta_S$ for some real number $\mu > 0$, we obtain for this term (we regard Y as r-independent)

$$\begin{aligned} \int_{\partial B_r(0)} r^{-2}\varDelta_S u\ Y &= \int_{\partial B_1(0)} r^{-2}\varDelta_S[u(r\,\cdot)]\ Y\ r^{n-1} = \int_{\partial B_1(0)} r^{-2}[u(r\,\cdot)]\ \varDelta_S Y\ r^{n-1} \\ &= -\mu \int_{\partial B_1(0)} r^{-2}[u(r\,\cdot)]\ Y\ r^{n-1} = -\mu \int_{\partial B_r(0)} r^{-2} u\ Y\,. \end{aligned}$$

Using the definition of v, we arrive at

$$\begin{aligned} 0 &= \int_{\partial B_r(0)} r^{-(n-1)} \partial_r (r^{n-1} \partial_r u)\, Y - \mu \int_{\partial B_r(0)} r^{-2} u\, Y + \int_{\partial B_r(0)} \lambda u\, Y \\ &= r^{-(n-1)} \partial_r (r^{n-1} \partial_r v)(r) + \left(\lambda - \frac{\mu}{r^2}\right) v(r) \\ &= \partial_r^2 v(r) + \frac{n-1}{r} \partial_r v(r) + \left(\lambda - \frac{\mu}{r^2}\right) v(r)\,. \end{aligned}$$

This is an ordinary differential equation for v and it remains to analyze this equation.

In some sense, there appear three constants in this equation, namely λ, μ and $n-1$. We first note that the factor λ can be transformed easily into the factor 1: We switch from the old independent variable r to the new independent variable $s = r\sqrt{\lambda}$. This introduces a factor λ in all terms except for the term λv. We obtain for $V(s) := v(r)$ the equation with the factor 1 instead of λ:

$$\partial_s^2 V(s) + \frac{n-1}{s} \partial_s V(s) + \left(1 - \frac{\mu}{s^2}\right) V(s) = 0\,. \tag{2.5}$$

For the pre-factor $n-1=1$, this is Bessel's differential equation with parameter μ.

Step 3: Transforming the equation for V into an equation for W. As we will see next, also the parameter $n-1$ in the equation can be transformed. This is also what Rellich does in his work, he transforms the parameter $n-1$ to the parameter 1 to obtain a standard Bessel equation. He can then use the knowledge on Bessel functions. Since we want a self-contained proof, we deviate here from the original proof and transform differently, aiming at the pre-factor 0 instead of $n-1$.

We use the new unknown $W(s) := V(s) s^{(n-1)/2}$. The derivatives are

$$\begin{aligned} \partial_s W(s) &= \partial_s V(s) s^{(n-1)/2} + \frac{n-1}{2} V(s) s^{(n-3)/2}\,, \\ \partial_s^2 W(s) &= \partial_s^2 V(s) s^{(n-1)/2} + (n-1) \partial_s V(s) s^{(n-3)/2} + \frac{(n-1)(n-3)}{4} V(s) s^{(n-5)/2}\,. \end{aligned}$$

Using the fact that, up to the factor $s^{(n-1)/2}$, the first two terms in the last line coincide with the first two terms in (2.5), we obtain a simple equation for W:

$$\begin{aligned} &\partial_s^2 W(s) + W(s) \\ &\quad = -\left(1 - \frac{\mu}{s^2}\right) V(s) s^{(n-1)/2} + \frac{(n-1)(n-3)}{4} V(s) s^{(n-5)/2} + V(s) s^{(n-1)/2} \\ &\quad = \frac{\mu}{s^2} V(s) s^{(n-1)/2} + \frac{(n-1)(n-3)}{4} V(s) s^{(n-5)/2} = \frac{\nu}{s^2} W(s) \end{aligned}$$

for the coefficient $\nu = \mu + (n-1)(n-3)/4$. This is a linear differential equation for W, it can be regarded as a perturbation of a harmonic oscillator. An important property is derived in the subsequent Lemma 2.2, namely the asymptotic behavior of solutions: $W(s) = \alpha \cos(s+\beta) + O(1/s)$ as $s \to \infty$ for some β and some $\alpha \neq 0$.[1]

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[1] It is at the corresponding point in his proof that Rellich writes "Bekanntlich ...", which means "it is well-known that ...". This was certainly true in his time. Nowadays, it is no longer standard to know the properties of Bessel functions.

The asymptotics of W imply $V(s) = \alpha s^{-(n-1)/2} \cos(s+\beta) + O(s^{-(n+1)/2})$ and hence $V(s)^2 s^{n-1} = |\alpha|^2 \cos^2(s+\beta) + O(s^{-2})$. Since this provides (2.4), the proof is complete. $\square$

Lemma 2.2 (Asymptotic behavior of a perturbed harmonic oscillator)**.** *We consider intervals* $(s_0, \infty) \subset \mathbb{R}$ *for* $s_0 > 0$. *Let* $W : [s_0, \infty) \to \mathbb{R}$ *be a solution of*

$$(\partial_s^2 + 1) W(s) = g(s) W(s) \qquad \forall s \in (s_0, \infty) \tag{2.6}$$

for some function g *with the bound* $|g(s)| \le C_0 s^{-2}$. *Then* W *is bounded and has, for some* $\alpha, \beta, C_1 \in \mathbb{R}$, *the asymptotic behavior*

$$|W(s) - \alpha \cos(s+\beta)| \le C_1 s^{-1} \,. \tag{2.7}$$

When W *is not vanishing identically, there holds* $\alpha \neq 0$.

Proof. Variation of constants. The proof is based on the variation of constants formula. In order to write (2.6) as a first order system, we set

$$X(s) := \begin{pmatrix} W(s) \\ W'(s) \end{pmatrix}, \quad A := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad F(s) := \begin{pmatrix} 0 \\ g(s)W(s) \end{pmatrix},$$

such that (2.6) reads

$$\partial_s X(s) + A \cdot X(s) = F(s) \,.$$

The fundamental solution of $\partial_s X + A \cdot X = 0$ is the matrix $S(t) = \exp(-At)$, it contains only entries $\pm \sin(t)$ and $\cos(t)$. Variations of constants is to write the solution $X(t)$, for arbitrary t_0 and t, as

$$X(t) = S(t-t_0) X(t_0) + \int_{t_0}^{t} S(t-s) F(s) \, ds \,. \tag{2.8}$$

A simple differentiation shows the correctness of this formula.

Boundedness. For arbitrary time instances $t_0 < t_1$, we set $C_X := \sup_{t_0 \le t \le t_1} |X(t)|$ and $\eta = \int_{t_0}^{\infty} s^{-2} \, ds = 1/t_0$. Because of $|F(s)| \le C_0 s^{-2} C_X$ and $|S(s)| \le 1$, we obtain from (2.8) the estimate $|X(t) - S(t-t_0)X(t_0)| \le C_0 \eta C_X \le C_X/2$, where the last inequality holds for every $t_0 > 2C_0$. The estimate is independent of t_1. Since $S(t-t_0)X(t_0)$ is bounded, independent of t_1, we conclude

$$C_X = \sup_{t_0 \le t \le t_1} |X(t)| \le |X(t_0)| + C_X/2 \,.$$

We keep t_0 fixed and conclude $C_X \le 2|X(t_0)|$, independent of t_1. Having C_X independent of t_1 implies that $X(t)$ is bounded, independent of $t < \infty$.

Decay and limit formula. With the same arguments we can now improve the estimates. We choose a sequence $t_k \to \infty$ and consider $t_0 = t_k$ in (2.8). The vector $X(t_k)$ determines $\alpha_k \in [0, \infty)$ and $\beta_k \in [0, 2\pi)$ such that

$$S(t_k - s) X(t_k) = \begin{pmatrix} \alpha_k \cos(s+\beta_k) \\ -\alpha_k \sin(s+\beta_k) \end{pmatrix} \qquad \forall s > s_0 \,.$$

Since α_k is bounded, we find a subsequence and limits such that $\alpha_k \to \alpha \geq 0$ and $\beta_k \to \beta \in [0, 2\pi]$. This implies that the first part of the solution in (2.8) converges in the desired way: The first component satisfies $e_1 \cdot S(t_k - s)X(t_k) \to \alpha \cos(s + \beta)$ for all s.

The other term satisfies, using the boundedness of the solution, $|X(s)| \leq C_X$ (possibly with a new constant C_X), for every fixed t:

$$\varepsilon_k(t) := \int_{t_k}^{t} S(t-s)F(s)\, ds\,, \qquad |\varepsilon_k(t)| \leq \int_{t}^{t_k} C_0 C_X s^{-2}\, ds \leq C_0 C_X t^{-1}\,.$$

By (2.8), we can write the solution as

$$\begin{aligned} W(t) = e_1 \cdot X(t) &= e_1 \cdot S(t - t_k)X(t_k) + e_1 \cdot \int_{t_k}^{t} S(t-s)F(s)\, ds \\ &= \alpha_k \cos(t + \beta_k) + e_1 \cdot \varepsilon_k(t)\,. \end{aligned}$$

Taking the limit $k \to \infty$, we obtain (2.7). We furthermore obtain an integral formula for W: With $F(s) = g(s)W(s)\, e_2$, there holds, for every t,

$$W(t) = \alpha \cos(t + \beta) + e_1 \cdot \int_{\infty}^{t} S(t-s)F(s)\, ds\,. \tag{2.9}$$

It remains to prove the last statement concerning $\alpha \neq 0$. We will show: When (2.7) holds with $\alpha = 0$, then W is vanishing identically. We therefore now assume $\alpha = 0$. We choose $t_0 \geq 2C_0$ and use $C_X := \sup_{t_0 \leq t < \infty} |W(t)|$. Then (2.9) implies, for arbitrary $t \geq t_0$,

$$|W(t)| \leq \int_{\infty}^{t} |S(t-s)|\, |F(s)|\, ds \leq \int_{\infty}^{t} C_0 s^{-2} C_X\, ds \leq t_0^{-1} C_0 C_X \leq C_X/2\,.$$

Taking the supremum over $t \geq t_0$, we obtain $C_X \leq C_X/2$. This implies $C_X = 0$ and hence $W \equiv 0$ on $[t_0, \infty)$. As a solution of a linear ordinary differential equation, with vanishing boundary data in t_0, the function W must vanish identically. □

3 Basic facts on spherical harmonics

We collect some important facts that are related to the Laplace-Beltrami operator and to spherical harmonics. We follow the exposition of [2], where more details on the construction can be found. We consider the space $L^2(S^{n-1})$ of square integrable functions on the unit sphere in $\mathbb{R}^n$. For a smooth function $u : S^{n-1} \to \mathbb{R}$ and a point $x \in S^{n-1}$, we want to define the (tangential) gradient $g = \nabla_T u(x) \in \mathbb{R}^n$. We demand that g is a tangential vector, $x \cdot g = 0$ with scalar product in $\mathbb{R}^n$. We consider a differentiable curve $\gamma : (-\varepsilon, \varepsilon) \to S^{n-1}$ with $\gamma(0) = x$ and $\gamma'(0) = v$. In order to have the chain rule, we *define* $g = \nabla_T u(x)$ as the vector with the property $\partial_\varepsilon(u \circ \gamma)(0) = g \cdot v$. This determines $g = \nabla_T u(x)$.

When u is defined in a neighborhood of a point $x \in S^{n-1} \subset \mathbb{R}^n$, then $\nabla_T u(x)$ is actually the same as the projection of the full gradient $\nabla u(x)$ onto the tangent space. This is a consequence of the chain rule for $u \circ \gamma$.

For two smooth functions $u, \varphi : S^{n-1} \to \mathbb{R}$, we can define the expression

$$a(u,\varphi) := \int_{S^{n-1}} \{ u\,\varphi \ + \nabla_T u \cdot \nabla_T \varphi \}\ . \tag{3.1}$$

The space $H^1(S^{n-1})$ is the closure of $C^1(S^{n-1},\mathbb{R})$ with respect to $\|u\|^2_{H^1} := a(u,u)$. It is a Hilbert space by construction. Since $a : H^1 \times H^1 \to \mathbb{R}$ is a scalar product and since $H^1(S^{n-1}) \subset L^2(S^{n-1})$ is compact, there exists a complete family of eigenfunctions $(\Phi_j)_{j\in\mathbb{N}}$ satisfying $a(\Phi_j,\varphi) = \lambda_j \langle \Phi_j, \varphi\rangle_{L^2}$ for all $\varphi \in H^1(S^{n-1})$. Idea of this fact, which is related to a spectral theorem: Minimize $a(u,u)$ among all u with $\|u\|_{L^2(S^{n-1})} = 1$ to find a first eigenfunction. Continue by minimizing in the subspace that is L^2-orthogonal to all the eigenfunctions that are already found.

Normalizing, we achieve $\|\Phi_j\|_{L^2} = 1$ and $a(\Phi_j,\Phi_j) = \lambda_j$ with the orthogonalities $\langle \Phi_i, \Phi_j\rangle_{L^2} = 0$ and $a(\Phi_i,\Phi_j) = 0$ for all $j \neq i$. The functions Φ_j are called spherical harmonics. They form a basis for $L^2(S^{n-1})$ and for $H^1(S^{n-1})$.

The Laplace-Beltrami operator is defined by the relation

$$\langle \Delta_S u, \varphi\rangle_{L^2(S^{n-1})} = -\int_{S^{n-1}} \nabla_T u \cdot \nabla_T \varphi\,. \tag{3.2}$$

for all $u,\varphi \in H^1(S^{n-1})$. Its relation to the classical Laplace operator is given, for $u \in C^2(\mathbb{R}^n,\mathbb{R})$, by the formula

$$\Delta u = \frac{1}{r^{n-1}}\partial_r(r^{n-1}\partial_r)u + \frac{1}{r^2}\Delta_S u\,, \tag{3.3}$$

where the last term must be interpreted with care. We sketch the argument leading to (3.3). For arbitrary smooth functions u,φ where φ has compact support, we calculate with $S := S^{n-1}$:

$$\begin{aligned}
-\int_{\mathbb{R}^n} \Delta u \cdot \varphi &= \int_{\mathbb{R}^n} \nabla u \cdot \nabla\varphi = \int_0^\infty \int_{rS} \nabla u \cdot \nabla \varphi \, dS\, dr \\
&= \int_0^\infty \int_{rS} \partial_r u \cdot \partial_r \varphi + \nabla_T u \cdot \nabla_T \varphi \, dS\, dr \\
&= \int_0^\infty \int_S \partial_r u(r\xi) \cdot \partial_r \varphi(r\xi) dS(\xi)\, r^{n-1}\, dr \\
&\quad + \int_0^\infty \int_S r^{-1}\nabla_T(u(r\ .))(\xi) r^{-1} \nabla_T(\varphi(r\ .))(\xi) dS(\xi)\, r^{n-1}\, dr \\
&= -\int_0^\infty \int_S \frac{1}{r^{n-1}}\partial_r(r^{n-1}\partial_r u(.\ \xi))(r)\varphi(r\xi) dS(\xi)\, r^{n-1}\, dr \\
&\quad - \int_0^\infty \int_S \frac{1}{r^2}\Delta_S[u(r\ .)](\xi)\varphi(r\xi) dS(\xi)\, r^{n-1}\, dr\,.
\end{aligned}$$

This yields (3.3) and makes clear how the last term has to be interpreted.

Literatur

[1] Franz Rellich. über das asymptotische Verhalten der Lösungen von $\Delta u + \lambda u = 0$ in unendlichen Gebieten. *Jber. Deutsch. Math.-Verein.*, 53:57–65, 1943.

[2] Ben Schweizer. *Partielle Differentialgleichungen.* Springer-Verlag, Berlin, 2013. Eine anwendungsorientierte Einführung. [An application-oriented introduction].