Lecture notes:
The Lions-Nečas Lemma

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Abstract: The Lions-Nečas Lemma is a fundamental result in the theory of Partial Differential Equations. Our aim is to provide a complete proof of this important result in modern mathematical language. We essentially follow the proof of Nečas.

1 Introduction

We are interested in the following result. It considers Lipschitz domains in $\mathbb{R}^n$ for arbitrary dimension $n \in \mathbb{N}$. We write $H^{-1}(\Omega)$ for the dual space of $H_0^1(\Omega)$.

Lemma 1.1 (Lions-Nečas Lemma)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, the dimension is $n \in \mathbb{N}$. Then, for a constant $C = C(\Omega) > 0$, there holds*

$$\|u\|_{L^2(\Omega)} \leq C \left(\|\nabla u\|_{H^{-1}(\Omega)} + \|u\|_{H^{-1}(\Omega)} \right) \qquad \forall u \in L^2(\Omega)\,. \tag{1.1}$$

The lemma is stated and proved in [2], Chapter 3, Lemma 7.1.

1.1 Comments on the statement

Names. An early observation of Lions is that, for sufficiently smooth domains Ω, when $u \in H^{-1}(\Omega)$ has a distributional gradient $\nabla u \in H^{-1}(\Omega)$, then there holds $u \in L^2(\Omega)$. This explains why the name Lions is associated with the above lemma. An early proof of the result for smooth domains was given by Magenes and Stampacchia. The first proof for Lipschitz-domains was given by Nečas. Because of this history, all of the names Lions, Magenes, Stampacchia, Nečas are attributed to this important result.

Relation to other results. The result is presented as Lemma 23.5 in [3]. In that text book, the lemma is used to derive three other results: (1) The image of the gradient operator $L^2(\Omega,\mathbb{R}) \to H^{-1}(\Omega,\mathbb{R}^n)$ is closed (Lemma 23.8). (2) The divergence operator $H_0^1(\Omega,\mathbb{R}^n) \to L^2(\Omega,\mathbb{R})$ possesses a continuous right-inverse (Korollar 23.9). (3) Korn's inequality (Theorem 25.5). This list shows the strength of Lemma 1.1.

Above, four results are listed: (0) Lions-Nečas Lemma 1.1, (1) image of the gradient operator gradient is closed, (2) Bogovskiĭ-operator or Babuska-Aziz inequality, (3) Korn's inequality. Indeed, the four results are actually equivalent in the sense that there are simple proofs that derive one of the above facts from any other of the above facts. This is discussed in [1, 4].

Stronger statement. The statement of Lemma 1.1 can be sharpened in the spirit of the original Lions-lemma: Every $u \in H^{-1}(\Omega)$ with $\nabla u \in H^{-1}(\Omega)$ has the property $u \in L^2(\Omega)$.

This can be deduced from (1.1), but, in our opinion, the proof is not trivial. The statement follows immediately when we find, for u as above, a sequence $u_k \in L^2(\Omega)$ with $u_k \to u$ in $H^{-1}(\Omega)$ such that also $\nabla u_k \to \nabla u$ in $H^{-1}(\Omega)$. For the construction of the sequence u_k: Use a partition of unity, flatten the boundary pieces with domain parametrizations, use approximating sequences for half-spaces and exploit the estimates of Steps 2 and 3 of our proof below.

2 Proof of the Lions-Nečas Lemma

2.1 On the proof for cubes

The proof on cubes can be performed with Fourier series and can be found in many references, one is [3], where the proof is given for $\Omega = (0,\pi)^n$. We collect the essential steps here.

Since one must use $H^1_0(\Omega)$-test-functions to calculate $H^{-1}(\Omega)$-norms, it is convenient to use the extended cube $\tilde{\Omega} = (-\pi,\pi)^n$, and to identify any function $u : \Omega \to \mathbb{R}$ with its odd extension to $\tilde{\Omega}$. Working in the class of odd functions on $\tilde{\Omega}$ has the following advantage: The restriction of an arbitrary odd function of class $H^1(\tilde{\Omega})$ satisfies boundary conditions and is of class $H^1_0(\Omega)$.

A function on Ω has the form

$$u(x) = \sum_{k\in\mathbb{Z}^n} \hat{u}(k) e^{ik\cdot x},$$

and gradients can be expressed accordingly with prefactors ik. The elementary norms are

$$\|u\|^2_{L^2(\tilde{\Omega})} = C_1 \sum_{k\in\mathbb{Z}^n} |\hat{u}(k)|^2, \qquad \|\nabla u\|^2_{L^2(\tilde{\Omega})} = C_1 \sum_{k\in\mathbb{Z}^n} |k|^2 |\hat{u}(k)|^2,$$

with $C_1 = (2\pi)^n$. This allows to calculate, with $C_2 = C_1 2^{-n}$,

$$\|u\|^2_{H^{-1}(\Omega)} \le C_2 \sum_{k\in\mathbb{Z}^n} \frac{|\hat{u}(k)|^2}{1+|k|^2}.$$

Using the algebraic identity $1 = \frac{1}{1+|k|^2} + \frac{|k|^2}{1+|k|^2}$, we conclude

$$\begin{aligned}
\|u\|^2_{L^2(\Omega)} &= C_2 \sum_{k\in\mathbb{Z}^n} |\hat{u}(k)|^2 \\
&= C_2 \sum_{k\in\mathbb{Z}^n} \frac{1}{1+|k|^2} |\hat{u}(k)|^2 + C_2 \sum_{k\in\mathbb{Z}^n} \frac{1}{1+|k|^2} |k|^2 |\hat{u}(k)|^2 \\
&= \|u\|^2_{H^{-1}(\Omega)} + \|\nabla u\|^2_{H^{-1}(\Omega)}.
\end{aligned}$$

This is the desired inequality (1.1).

2.2 Proof for domains

We now give the proof of Lemma 1.1 for general bounded Lipschitz domains. We mention that a sketch of this proof is given in [3], we use the same notation here. The proof uses a non-elementary inequality on transformations, namely (2.3). The inequality follows easily for C^2-domains, but the proof for Lipschitz domains is much more complicated.

We postpone the proof of (2.3) to Section 3. This means that Section 2 contains the complete proof of Lemma 1.1 for C^2-domains. The proof for Lipschitz domains requires the analysis of Section 3, where (2.3) is verified.

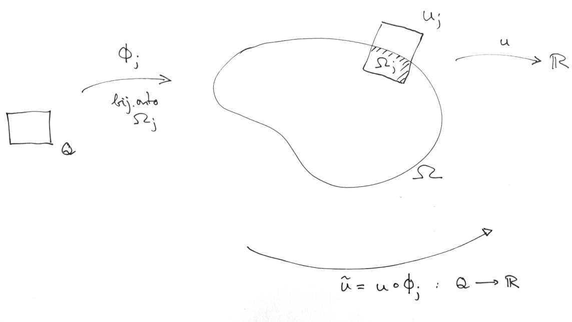

Abbildung 1: Illustration of the notation for the covering of Ω.

Step 1: Finite cover of $\partial\Omega$ and domain transformations. We cover Ω with open sets U_j, where $U_1, ..., U_m$ are cylinders that cover $\partial\Omega$, the set U_0 does not intersect the boundary. We may assume that every cylinder U_j has a base that is a cube in $\mathbb{R}^{n-1}$. For the pieces $\Omega_j := U_j \cap \Omega$ we can choose Lipschitz-continuous bijective maps (with the inverse map also Lipschitz),

$$\Phi_j : Q = (0, 2\pi)^n \to \Omega_j \,,$$

which map the upper boundary of Q to $U_j \cap \partial\Omega$. The parametrization can be used to transform functions on Ω_j to functions on Q as follows:

$$u : \Omega_j \to \mathbb{R} \,, \qquad \tilde{u} = u \circ \Phi_j : Q \to \mathbb{R} \,.$$

For an illustration see Figure 1.

The transformation $u \mapsto \tilde{u}$ defines a map $T_j : H_0^1(\Omega_j) \to H_0^1(Q)$. We use the same letter for the maps $T_j : L^2(\Omega_j) \to L^2(Q)$ and $T_j : H^1(\Omega_j) \to H^1(Q)$. We have to investigate the properties of $T_j : u \mapsto \tilde{u}$.

Step 2: Mapping properties in L^2 and H^1. The transformation formula yields

$$\|u\|_{L^2(\Omega_j)}^2 = \int_{\Omega_j} |u|^2 = \int_Q |\tilde{u}|^2 |\det(D\Phi_j)| \le C \|\tilde{u}\|_{L^2(Q)}^2 \,, \tag{2.1}$$

$$\|\nabla u\|_{L^2(\Omega_j)}^2 = \int_{\Omega_j} |\nabla(\tilde{u} \circ \Phi_j^{-1})|^2 = \int_{\Omega_j} |(\nabla\tilde{u}) \circ \Phi_j^{-1} D(\Phi_j^{-1})|^2 \le C \|\nabla\tilde{u}\|_{L^2(Q)}^2 \,. \tag{2.2}$$

The same calculations can be performed in the opposite direction and we conclude that T_j and T_j^{-1} are bounded maps between L^2-spaces and also bounded maps between H^1-spaces.

Step 3: Properties of the transformations in H^{-1}. We claim that good mapping properties are also available for H^{-1}-spaces. The proof is simple for C^2-domains and more involved for Lipschitz domains. In both cases, one has to estimate a term that involves second derivatives of the transformation, $D^2\Phi_j$. We will discuss and use the inequality

$$\big\| |D^2\Phi_j|\, \tilde\psi \big\|_{L^2(Q)} \le C \,\|\tilde\psi\|_{H^1(Q)} \qquad \forall \tilde\psi \in H^1_0(Q)\,. \tag{2.3}$$

The constant C in this inequality may depend on Ω_j and Φ_j.

For Φ_j of class C^2, it is clear that (2.3) holds for some constant C, since $|D^2\Phi_j|$ is bounded.

We now use (2.3) to verify properties of the transformation in H^{-1}-spaces. We calculate with the definition of H^{-1} as the dual space of H^1_0 and take the supremum over test-functions $\tilde\psi = T_j\psi \in H^1_0(Q)$ with $\|\tilde\psi\|_{H^1(Q)} \le 1$. We find for $\tilde u = T_j u$

$$\begin{aligned}
\|\tilde u\|_{H^{-1}(Q)} &= \sup_{\tilde\psi} \int_Q \tilde u\, \tilde\psi = \sup_{\tilde\psi} \int_{\Omega_j} u\, \psi \det(D\Phi_j^{-1}) \\
&\le \|u\|_{H^{-1}(\Omega_j)} \sup_{\tilde\psi} \|\psi \det(D\Phi_j^{-1})\|_{H^1(\Omega_j)} \\
&\le C \|u\|_{H^{-1}(\Omega_j)} \sup_{\tilde\psi} \|\tilde\psi \det(D\Phi_j^{-1}) \circ \Phi_j\|_{H^1(Q)} \overset{(2.3)}{\le} C \|u\|_{H^{-1}(\Omega_j)}\,.
\end{aligned}$$

We exploited that the H^1-norms in the domains can be compared with each other. The calculation can be made also with the inverse map and we conclude that the operators T_j and T_j^{-1} are bounded maps between H^{-1}-spaces.

We claim that also another property is available. We can calculate for $\tilde u = T_j u = u \circ \Phi_j$ and for arbitrary test-functions $\tilde\psi = T_j\psi \in H^1_0(Q)$ with $\|\tilde\psi\|_{H^1(Q)} \le 1$:

$$\begin{aligned}
\|\nabla \tilde u\|_{H^{-1}(Q)} &= \sup_{\tilde\psi} \int_Q \nabla \tilde u\, \tilde\psi = \sup_{\tilde\psi} \int_Q (\nabla u) \circ \Phi_j\, D\Phi_j\, \tilde\psi \\
&= \sup_{\tilde\psi} \int_{\Omega_j} \nabla u \cdot (D\Phi_j \circ \Phi_j^{-1} \,\det D\Phi_j^{-1})\, \psi \\
&\le \|\nabla u\|_{H^{-1}(\Omega_j)} \sup_{\tilde\psi} \left\| (D\Phi_j \circ \Phi_j^{-1} \,\det D\Phi_j^{-1})\, \psi \right\|_{H^1(\Omega_j)} \\
&\le C \|\nabla u\|_{H^{-1}(\Omega_j)} \sup_{\tilde\psi} \left\| D\Phi_j \,\det(D\Phi_j^{-1} \circ \Phi_j)\, \tilde\psi \right\|_{H^1(Q)} \overset{(2.3)}{\le} C \|\nabla u\|_{H^{-1}(\Omega_j)}\,.
\end{aligned}$$

We have therefore obtained, for arbitrary $u \in L^2(\Omega_j)$ and $\tilde u = T_j u$:

$$\|\nabla \tilde u\|_{H^{-1}(Q)} = \|(\nabla u \circ \Phi_j) D\Phi_j\|_{H^{-1}(Q)} \le C \|\nabla u\|_{H^{-1}(\Omega_j)}\,. \tag{2.4}$$

Step 4: Conclusion. We now conclude the proof with the above properties of the transformations T_j. We choose a differentiable partition of unity $(\eta_j)_j$ subordinate to the covering $(U_j)_j$. We consider $u \in L^2(\Omega)$ and calculate with constants C that may change from one line to the next. The function $u\eta_0$ on the inner domain U_0 is also a function on a large cube (containing U_0), this cube can be parametrized with a linearly affine map Φ_0 over Q

and the above estimates for norms hold also for $u\eta_0$. We use (1.1) on the cube Q, where this estimate is already shown.

$$\begin{aligned}
\|u\|_{L^2(\Omega)} &= \left\| \sum_j u\eta_j \right\|_{L^2(\Omega)} \le \sum_j \left\|u\eta_j\right\|_{L^2(\Omega_j)} \le C \sum_j \|T_j(u\eta_j)\|_{L^2(Q)} \\
&\overset{(1.1)}{\le} C \sum_j \left(\|\nabla(T_j(u\eta_j))\|_{H^{-1}(Q)} + \|T_j(u\eta_j)\|_{H^{-1}(Q)} \right) \\
&\overset{(2.4)}{\le} C \sum_j \left(\|\nabla(u\eta_j)\|_{H^{-1}(\Omega_j)} + \|u\eta_j\|_{H^{-1}(\Omega_j)} \right) \\
&\le C \left(\|\nabla u\|_{H^{-1}(\Omega)} + \|u\|_{H^{-1}(\Omega)} \right).
\end{aligned}$$

In the last inequality we exploited, regarding the first term, that, for $\psi \in H_0^1(\Omega)$,

$$\int \nabla(u\eta_j)\psi = -\int u\eta_j \nabla\psi = -\int u\nabla(\eta_j\psi) + \int u\psi\nabla\eta_j = \int \nabla u(\eta_j\psi) + u\psi\nabla\eta_j\,.$$

Together with the fact that the H^1-norms of test-functions $\eta_j\psi$ can be estimated by the H^1-norms of ψ, we obtain the last inequality. We have thus obtained the claim of Lemma 1.1. We recall that we have used (2.3), which is clear only for C^2-bounded domains.

3 Proof of inequality (2.3) for Lipschitz domains

The inequality (2.3) is verified for Lipschitz domains when we can show a pointwise inequality of $D^2\Phi_j$ and a corresponding weighted L^2-inequality for H_0^1-functions, more precisely

$$|D^2\Phi_j(x)| \le \frac{C}{|x_n|} \qquad \forall x \in Q\,, \tag{3.1}$$

$$\int_Q \frac{1}{|x_n|^2} |\tilde{\psi}(x)|^2\,dx \le C\|\tilde{\psi}(x)\|^2_{H_0^1(Q)}\,. \tag{3.2}$$

Here, Q is a bounded domain of the form $Q = S \times (-H, 0)$ with upper boundary $S \times \{0\} = \{x_n = 0\}$.

Inequality (3.2) is provided by the Hardy inequality, see (3.4) of Lemma 3.2. Inequality (3.1) is valid for an appropriate choice of a parametrization of the Lipschitz domain, see Lemma 3.4.

3.1 Hardy inequality

We need only a quite elementary Hardy inequality. We show it first in one space dimension and then transfer the result to arbitrary dimension.

Lemma 3.1 (Simple one-dimensional Hardy inequality)**.** *Let $F : \Omega \to \mathbb{R}$ be a function on the interval $\Omega = (0,L) \subset \mathbb{R}$, of class $F \in C_c^1(\Omega,\mathbb{R})$. Then there holds*

$$\int_\Omega \frac{1}{x^2}|F(x)|^2\,dx \le 4\int_\Omega |F'(x)|^2\,dx\,. \tag{3.3}$$

Proof. We use $f = F'$ and, in this order: (i) integration by parts with $\partial_x(1/x) = -(1/x^2)$. (ii) Cauchy-Schwarz.

$$\int_\Omega \frac{1}{x^2}|F(x)|^2\,dx = \int_\Omega \frac{1}{x} 2F(x)f(x)\,dx \le 2\left(\int_\Omega \frac{1}{x^2}|F(x)|^2\,dx\right)^{1/2}\left(\int_\Omega |f(x)|^2\,dx\right)^{1/2}.$$

We may assume that the left-hand side is not vanishing since otherwise the claim is clear. Dividing by the square root of the left-hand side and then taking the square of the result, we obtain (3.3). □

Lemma 3.2 (Simple Hardy inequality in arbitrary dimension)**.** *We consider* $1 \le n \in \mathbb{N}$ *and* $L > 0$ *and the domain* $Q = \mathbb{R}^{n-1} \times (0,L) \subset \mathbb{R}^n$. *Then, for an arbitrary function* $u \in H_0^1(Q,\mathbb{R})$, *we have*

$$\int_Q \frac{1}{x_n^2}|u(x)|^2\,dx \le 4\int_Q |\nabla u(x)|^2\,dx. \tag{3.4}$$

Proof. An arbitrary function $u \in H_0^1(Q,\mathbb{R})$ can be approximated (in H^1) by a sequence of functions $u_k \in C_c^1(Q,\mathbb{R})$. The inequality allows to take the limit $k \to \infty$ (the lim-inf on the left-hand side), it is therefore sufficient to show (3.4) for $u \in C_c^1(Q,\mathbb{R})$.

For arbitrary $x' \in \mathbb{R}^{n-1}$, we consider the function $F(x_n) := u((x',x_n))$ and apply (3.3). This provides

$$\int_Q \frac{1}{x_n^2}|u((x',x_n))|^2\,dx_n \le 4\int_Q |\nabla u((x',x_n))|^2\,dx_n.$$

An integration over $x' \in \mathbb{R}^{n-1}$ provides (3.4). □

3.2 Parametrizations with estimates for second derivatives

The Lipschitz parametrization of Lipschitz domains with good properties for all derivatives is based on the following result.

Lemma 3.3 (Extension with estimates)**.** *Let* $g : \mathbb{R}^{n-1} \to \mathbb{R}$ *be Lipschitz-continuous with constant* C_L. *We consider the function that is defined, for an arbitrary point* $x = (x',-h) \in Q = \mathbb{R}^{n-1} \times (-\infty,0)$, *as the weighted average*

$$u(x',-h) = \frac{1}{\kappa h^{n-1}}\int_{B_h(x')} g(y)\exp\left(-\frac{|x'-y|^2}{h^2-|x'-y|^2}\right)dy, \tag{3.5}$$

with the positive constant $\kappa > 0$ *defined below, see* (3.10). *Then* u *is continuous on* $\bar{Q}$ *and an extension of* g *in the sense that* $u(x',0) = g(x')$ *for every* $x' \in \mathbb{R}^{n-1}$. *Furthermore, for a constant* $C > 0$ *that depends only on* C_L, *there holds*

$$|Du(x)| \le C \quad \text{and} \quad \left|D^2u(x))\right| \le \frac{C}{h} \qquad \forall x = (x',-h) \in Q. \tag{3.6}$$

With extensions u as in Lemma 3.3, we can easily define the parametrization with optimal bounds for derivatives.

Lemma 3.4 (Parametrization of Lipschitz domains with bounds for first and second derivatives)**.** *We consider a Lipschitz-function* $g : \mathbb{R}^{n-1} \to \mathbb{R}$*. Then, the domain*

$$\Omega := \left\{ (x', x_n) \,\middle|\, x' \in \mathbb{R}^{n-1}\,,\ x_n \in \mathbb{R}\,,\ x_n < g(x') \right\} \tag{3.7}$$

can be parametrized over $Q = \mathbb{R}^{n-1} \times (-\infty, 0)$ *with a Lipschitz map* $\Phi : Q \to \Omega$ *that is invertible with Lipschitz inverse. Furthermore, for some* $C > 0$ *that depends only on the Lipschitz constant of* g*:*

$$\left| D^2 \Phi((x', x_n)) \right| \le \frac{C}{|x_n|} \qquad \forall (x', x_n) \in Q\,. \tag{3.8}$$

Proof. We use the extension u of Lemma 3.3 of the height function g. With the help of u, we can define the desired parametrization. We set, for a large constant $M \in \mathbb{R}_+$,

$$\Phi : Q \to \Omega\,, \quad (x', x_n) \mapsto (x', M x_n + u(x', x_n))\,. \tag{3.9}$$

It remains to check the properties of Φ. Since u has bounded first derivatives, by choosing M large, we obtain $\partial_n \Phi_n(x) = M + \partial_n u(x) > 0$. This implies that Φ is invertible with positive determinant of $D\Phi$. By the inverse function theorem, the inverse is also Lipschitz continuous.

The bound for second derivatives follows from those for u, see (3.6). □

3.3 Proof of Lemma 3.3

We start with the investigation of some integrals. For the vertical coordinate, we write $h = -x_n = |x_n| > 0$. We study certain integrals over balls $B_h(0) \subset \mathbb{R}^{n-1}$. The substitution $y = hz$ allows to calculate

$$\begin{aligned} \frac{1}{h^{n-1}} \int_{B_h(0)} \exp\left(-\frac{|y|^2}{h^2 - |y|^2} \right) dy &= \int_{B_1(0)} \exp\left(-\frac{|hz|^2}{h^2 - |hz|^2} \right) dz \\ &= \int_{B_1(0)} \exp\left(-\frac{|z|^2}{1 - |z|^2} \right) dz =: \kappa\,. \end{aligned} \tag{3.10}$$

In particular, the number κ does not depend on h. We see that, indeed, averages of g are calculated with the extension of (3.5). This shows that u is continuous on $\bar{Q}$ and has the boundary values g.

Almost the same calculation can be used also for a more singular integrand. For $m \ge 0$, $m \in \mathbb{N}$, we observe

$$\begin{aligned} &\frac{1}{h^{n-1}} \int_{B_h(0)} \exp\left(-\frac{|y|^2}{h^2 - |y|^2} \right) \left[\frac{1}{(h^2 - |y|^2)^m} \right] dy \\ &= \int_{B_1(0)} \exp\left(-\frac{|hz|^2}{h^2 - |hz|^2} \right) \left[\frac{1}{(h^2 - |hz|^2)^m} \right] dz \\ &= \frac{1}{h^{2m}} \int_{B_1(0)} \exp\left(-\frac{|z|^2}{1 - |z|^2} \right) \left[\frac{1}{(1 - |z|^2)^m} \right] dz \le \frac{C}{h^{2m}}\,. \end{aligned} \tag{3.11}$$

We additionally observe a fact on derivatives of the argument of the exponential. We calculate a derivative in direction y_j for $j \le n-1$ with the quotient rule:

$$\partial_j \left(\frac{|y|^2}{h^2 - |y|^2} \right) = \frac{h^2\, 2y_j}{(h^2 - |y|^2)^2}\,,$$

and the derivative with respect to h is

$$\partial_h \left(\frac{|y|^2}{h^2 - |y|^2} \right) = -\frac{|y|^2\, 2h}{(h^2 - |y|^2)^2}\,.$$

We can relate the two terms by writing

$$\sum_j y_j \partial_j \left(\frac{|y|^2}{h^2 - |y|^2} \right) = \frac{h^2\, 2|y|^2}{(h^2 - |y|^2)^2} = -h\, \partial_h \left(\frac{|y|^2}{h^2 - |y|^2} \right). \tag{3.12}$$

Proof of Lemma 3.3. Step 1: Estimates for first derivatives. The function u is defined as a convolution. Accordingly, a derivative ∂_j for $j \le n-1$ acting in the variable x' can also been applied to g (one way to write this is with an integration by parts). We have

$$\partial_j u(x', -h) = \frac{1}{\kappa h^{n-1}} \int_{B_h(x')} \partial_j g(y) \exp\left(-\frac{|x' - y|^2}{h^2 - |x' - y|^2} \right) dy\,. \tag{3.13}$$

This is a weighted average of $\partial_j g$ and we conclude that all derivatives $\partial_j u(x)$ are bounded, independent of x.

Let us calculate the vertical first derivative. Considering, without loss of generality, the point $x' = 0$, we find

$$\begin{aligned} \partial_h u(0, -h) = &-\frac{n-1}{h} \frac{1}{\kappa h^{n-1}} \int_{B_h(0)} g(y) \exp\left(-\frac{|y|^2}{h^2 - |y|^2} \right) dy \\ &- \frac{1}{\kappa h^{n-1}} \int_{B_h(0)} g(y) \exp\left(-\frac{|y|^2}{h^2 - |y|^2} \right) \partial_h \left(\frac{|y|^2}{h^2 - |y|^2} \right) dy\,. \end{aligned} \tag{3.14}$$

We rewrite the second term using (3.12) and with an integration by parts:

$$\begin{aligned} &- \frac{1}{\kappa h^{n-1}} \int_{B_h(0)} g(y) \exp\left(-\frac{|y|^2}{h^2 - |y|^2} \right) \partial_h \left(\frac{|y|^2}{h^2 - |y|^2} \right) dy \\ &= \frac{1}{\kappa h^{n-1}} \int_{B_h(0)} g(y) \exp\left(-\frac{|y|^2}{h^2 - |y|^2} \right) \frac{1}{h} \sum_j y_j \partial_j \left(\frac{|y|^2}{h^2 - |y|^2} \right) dy \\ &= -\frac{1}{h} \frac{1}{\kappa h^{n-1}} \int_{B_h(0)} g(y) \sum_j y_j \partial_j \exp\left(-\frac{|y|^2}{h^2 - |y|^2} \right) dy \\ &= \frac{n-1}{h} \frac{1}{\kappa h^{n-1}} \int_{B_h(0)} g(y) \exp\left(-\frac{|y|^2}{h^2 - |y|^2} \right) dy \\ &\quad + \frac{n-1}{h} \frac{1}{\kappa h^{n-1}} \int_{B_h(0)} \sum_j y_j \partial_j g(y) \exp\left(-\frac{|y|^2}{h^2 - |y|^2} \right) dy\,. \end{aligned}$$

When we insert this in (3.14), the first terms on the right hand sides cancel. We arrive at

$$\partial_h u(0,-h) = \frac{n-1}{h}\frac{1}{\kappa h^{n-1}}\int_{B_h(0)} y\cdot\nabla g(y)\exp\left(-\frac{|y|^2}{h^2-|y|^2}\right)dy\,. \tag{3.15}$$

Using $|y|<h$ and boundedness of $|\nabla g|$, we obtain boundedness of this expression. This concludes boundedness of all first derivatives as claimed in (3.6).

Step 2: Estimates for second derivatives. We calculate a second derivative with the help of (3.13), using a coordinate index $i\le n-1$. Afterwards, we insert the point $x'=0$ to find

$$\begin{aligned}\partial_i\partial_j u(0,-h) &= -\frac{1}{\kappa h^{n-1}}\int_{B_h(x')} \partial_j g(y)\partial_i \exp\left(-\frac{|y|^2}{h^2-|y|^2}\right)dy\\ &= \frac{1}{\kappa h^{n-1}}\int_{B_h(x')} \partial_j g(y)\exp\left(-\frac{|y|^2}{h^2-|y|^2}\right)\frac{h^2\,2y_i}{(h^2-|y|^2)^2}\,dy\,.\end{aligned}$$

We use that $|h^2\,2y_i|\le 2h^3$, boundedness of $|\partial_j g(y)|$ and (3.11) with $m=2$ to conclude

$$|\partial_i\partial_j u(0,-h)| \le C\frac{1}{h}\,. \tag{3.16}$$

This provides the bound of (3.6) on second tangential derivatives.

Let us consider a mixed derivative. Formula (3.15) for $\partial_h u$ can be written for an arbitrary argument x'. Differentiating with respect to an arbitrary direction $i\le n-1$ and evaluating again, without loss of generality, in $x'=0$, we find

$$|\partial_i\partial_h u(0,-h)| = \frac{n-1}{h}\frac{1}{\kappa h^{n-1}}\int_{B_h(0)} |\nabla g(y)|\left|\partial_i\left[y_i\exp\left(-\frac{|y|^2}{h^2-|y|^2}\right)\right]\right|dy\,. \tag{3.17}$$

With the same calculation as the one that was leading to (3.16) we conclude that also this second derivative is bounded as $|\partial_i\partial_h u(0,-h)|\le C/h$.

It remains to study second vertical derivatives. Formula (3.15) for $\partial_h u$ can be differentiated with respect to h. A first term is produced by the differentiation of h^{-n}, this term is bounded by C/h. A second term comes from the differentiation of the exponential. We find

$$|\partial_h^2 u(0,-h)| \le \frac{C}{h}+\frac{C}{h^n}\int_{B_h(0)} |y|\,\exp\left(-\frac{|y|^2}{h^2-|y|^2}\right)\left|\partial_h\left(-\frac{|y|^2}{h^2-|y|^2}\right)\right|dy\,.$$

The term $\partial_h(|y|^2/(h^2-|y|^2))$ has a nominator that is bounded by $2h|y|^2\le 2h^3$. We use (3.11) with $m=2$ and conclude that also the second term is bounded by C/h. This provides (3.6) for all second derivatives. □

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