Instability effects in hysteresis models for porous media flow

SIAM GS 2013 — Padova

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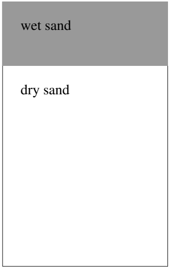
Fingering

Flow in a porous medium



Question

How does the water travel downwards ?

Experiments

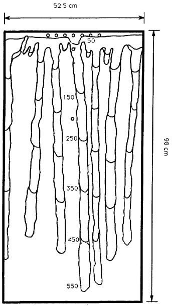
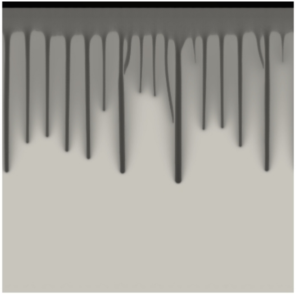


Fig. 5. Development of unstable wetting front in experiment 2.

From: Selker, Parlange, Steenhuis, Fingered Flow in Two Dimensions. Part 2. Predicting Finger Moisture Profile, 1992.

Numerics



Numerics by A. Rätz

The model includes static and dynamic hysteresis

Experimental observations

- We need a well-prepared medium: very dry sand
- fingers travel with constant speed
- the saturation profile is not monotone in x inside the finger

Mathematics

The standard Richards equation

- defines an L^1-contraction (Otto)
- L^2-stability (Duijn, Pieters, Raats)

Question:

How do we modify Richards equation to obtain fingering?

Modelling flow in porous media

Our aim is to describe the flow of water in unsaturated porous media.

Variables

domain	$\Omega \subset \mathbb{R}^N$
saturation	$s : \Omega \times (0,T) \to \mathbb{R}$
pressure	$p : \Omega \times (0,T) \to \mathbb{R}$
velocity	$v : \Omega \times (0,T) \to \mathbb{R}^N$

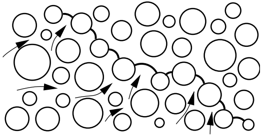
Equations

Darcy law	$v = -k(s)[\nabla p + e_x]$
conservation law	$\partial_t s + \nabla \cdot v = 0$
some relation	p to s

We combine these to the evolution equation

$$\partial_t s = \nabla \cdot (k(s)[\nabla p + e_x]).$$

(Standard) Pore Analysis

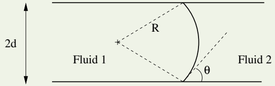


At a given saturation s, pores of radius $d_0(s)$ must be filled

Needs the pressure

$$p = p_c(s)$$

Tube-Model



d radius of the tube

θ contact angle

$\mathcal{H} = R^{-1}$ curvature

β surface tension

$p = \beta\mathcal{H} = F(d)$

Richards equation:

$$\partial_t s = \nabla \cdot (k(s)[\nabla p_c(s) + e_x])$$

Additional effect: Hysteresis

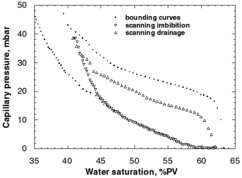
We better keep

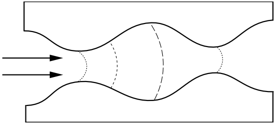
Richards equation with hysteresis

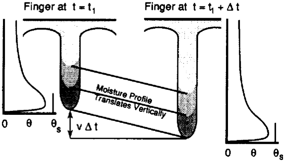
$\partial_t s = \nabla \cdot (k(s)[\nabla p + e_x])$ and a relation p to s

Hysteresis in porous media

… is important in fingering!







From: Selker, Parlange, Steenhuis, 1992

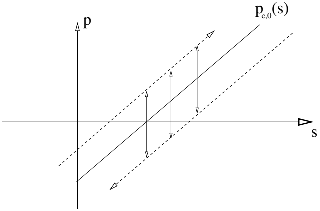
- Hassanizadeh and Gray, Thermodynamic basis of capillary pressure …, 1993
- Beliaev and Hassanizadeh, A theoretical model of hysteresis …, 2001

For fixed saturation s, demand $p \in [p_1, p_2] =: p_c(s) + [-\gamma, \gamma]$

Hysteresis model

$$\partial_t s = \nabla \cdot (k(s)[\nabla p + e_x])$$

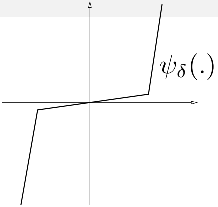
$$p \in p_c(s) + \gamma\, \mathbf{sign}(\partial_t s)$$



Existence results

Idea:
Discretize (h) and regularize (δ)

$$\partial_t s^{h,\delta} = \psi_\delta([p^{h,\delta} - p_c(s^{h,\delta})]/\gamma)$$
$$\Delta \tilde{p}^{h,\delta} = \psi_\delta([p^{h,\delta} - p_c(s^{h,\delta})]/\gamma)$$



Main task: s has time-regularity and p has space regularity.
Derive compactness from these facts!

Semi-linear problem, B.S. 2007

$$p \in p_c(s) + \gamma\, \mathbf{sign}(\partial_t s), \qquad \partial_t s = \Delta p$$

Fully non-linear problem, A.Lamacz, A.Rätz, B.S. 2011

$$p \in p_c(s) + \gamma\, \mathbf{sign}(\partial_t s) + \tau \partial_t s, \qquad \tau > 0$$
$$\partial_t s = \nabla \cdot (k(s)[\nabla p + e_x])$$

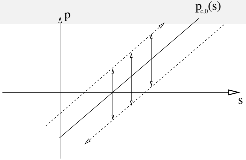
Two-phase flow, J.Koch, A.Rätz, B.S. 2013

$$p_1 - p_2 \in p_c(s) + \gamma\, \mathbf{sign}(\partial_t s) + \tau \partial_t s, \qquad \tau > 0$$
$$\partial_t s_j = \nabla \cdot (k_j(s_j)[\nabla p_j + g_j])$$

Homogenization result

Criticism of the model:

1. vertical scanning curves
2. “no loops”



Theorem [S. 2007]

Assume that many **play-type hysteresis** materials are homogenized.
Then: The evolution equation remains

$$\partial_t s = k^* \Delta p.$$

Homogenization leads to a **Prandtl-Ishlinskii hysteresis** relation,

$$s(x,t) = \int_I p_c^{-1}(w(x,y,t))\, dy, \qquad I = [0,1]$$

$$p(x) \in w(x,y) + \gamma(y)\, \mathbf{sign}(\partial_t w(x,y)) \quad \forall y \in I$$

Nonlinear homogenization result for two-phase flow in [P. Henning, M. Ohlberger, B.S.] M3AS, 2013

Can the model explain fingering?

Proposition (Stability)

Consider Richards equation with static hysteresis,

$$\partial_t s = \nabla \cdot (k(s)[\nabla p + e_x]) + f$$
$$p \in p_c(s) + \gamma\, \mathbf{sign}(\partial_t s)$$

We assume that **either** $\gamma = 0$ **or** *that* $k > 0$ *is independent of* s*. This system generates an* L^1*-contraction: For two solutions* s_j *and sources* f_j *there holds, for all* $t_2 > t_1$*,*

$$\int_\Omega |s_1 - s_2|(x, t_2)\, dx \leq \int_\Omega |s_1 - s_2|(x, t_1)\, dx + \int_{t_1}^{t_2} \int_\Omega |f_1 - f_2|(x, t)\, dx\, dt$$

Theorem (Instability) [S. 2012]

System is no L^1-contraction for $\gamma > 0$ and $k = k(s)$.

Proof of instability — one-dimensional analysis

Boundary condition: High pressure until $t = 0$, lower pressure afterwards

The switching pressure condition

- coincides with experimental description
- high saturation near upper boundary after short time

A free boundary problem: $X(t)$ and $q(t)$ free parameters

Right domain: hysteresis blocks evolution

$$k(s_1)[\partial_x p + 1] = q \qquad \text{on } \{(x,t) : X(t) < x < L_+\}$$

$$p(X(t)+0,t) = p_c(s_1(X(t))) + \gamma$$

$$p(L_+,t) = p_+$$

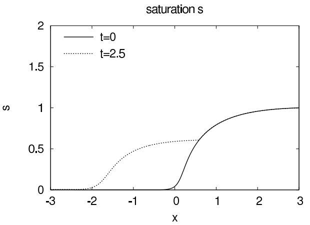
Left domain: standard Richards evolution

$$\partial_t s = \partial_x \left(k(s)[\partial_x p_c(s) + 1]\right) \qquad \text{on } \{(x,t) : x < X(t)\}$$

$$p(X(t)-0,t) = p_c(s_1(X(t))) + \gamma, \qquad p(L_-,t) = p_-$$

$$\left(k(s)[\partial_x p + 1]\right)(X(t)-0,t) = q$$

Proof of the instability theorem



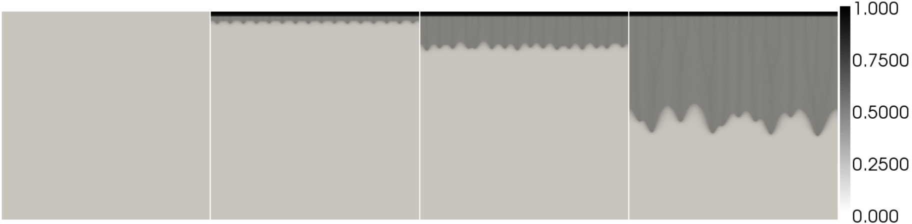
1. small perturbation of the initial values remains present for all times
2. below high saturation, the front travels faster — for all times

Conclusion: Richards equation with hysteresis and gravity is unstable

- Rigorous proof, based on a free boundary problem
- No heterogeneity of the medium assumed
- Instability for hysteresis and non-monotone boundary data

Onset of fingering

Two-dimensional numerical results for Richards equation:
static hysteresis, $\tau = 0$.



discrete saturations at $t = t_0 = -2$, $t \approx 509$, $t \approx 2508$, $t \approx 8487$.

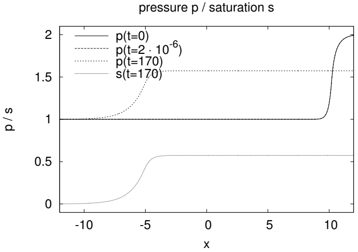
Result

static hysteresis *alone* can create an instability

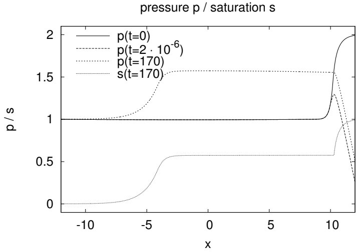
All numerical results by *A. Rätz, TU Dortmund*

Profiles in one space dimension, no dynamic factor

Pressure and saturation profiles **without dynamic term,** $\tau = 10^{-3}$
Time instances: $t = 0$, $t = 2 \cdot 10^{-6}$, $t = 170$



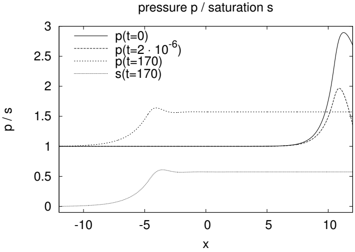
no static hysteresis
$\gamma = 0$, $p_c(s) = s + 1$



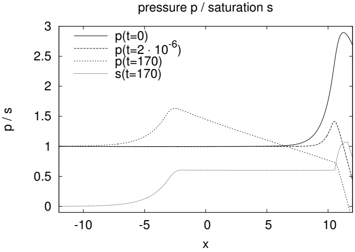
with static hysteresis
$\gamma = 1$, $p_c(s) = s$

Profiles in one space dimension, $\tau > 0$

Pressure and saturation profiles **with dynamic term,** $\tau = 5$
Time instances: $t = 0$, $t = 2 \cdot 10^{-6}$, $t = 170$



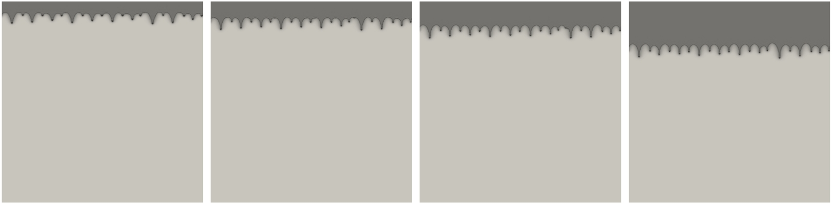
no static hysteresis
$\gamma = 0$, $p_c(s) = s + 1$



with static hysteresis
$\gamma = 1$, $p_c(s) = s$

Numerical results without static hysteresis

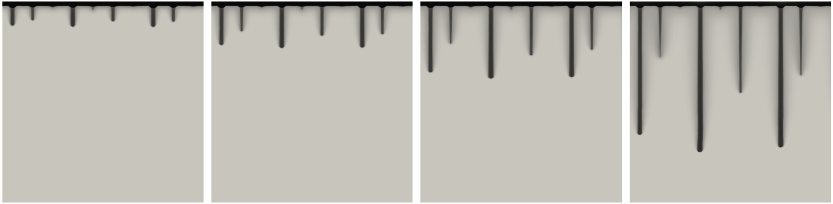
Evolution of saturation values for $\tau = 0.5$, no static hysteresis.



Richards equation, time instances $t \approx 56$, $t \approx 114$, $t \approx 201$, $t \approx 406$
deterministic perturbation of the initial values

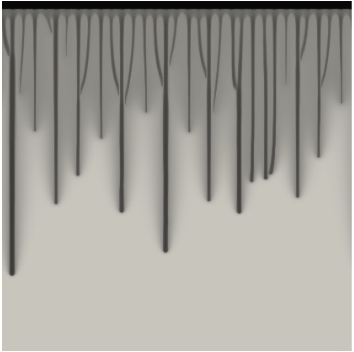
Numerical results with static hysteresis

Evolution of saturation values for $\tau = 0.5$ **with** static hysteresis



Richards equation, time instances $t \approx 56$, $t \approx 114$, $t \approx 201$, $t \approx 406$
deterministic perturbation of the initial values

Conclusions:



Fingering for Richards flow with hysteresis and dynamic effect

- hysteresis models for $\tau > 0$ are well-posed
- front solutions for hysteresis and $\tau = 0$ are unstable
- static hysteresis & $\tau > 0$ produces fingering

Thank you!