Energy method approach to existence results for the Helmholtz equation in periodic wave-guides

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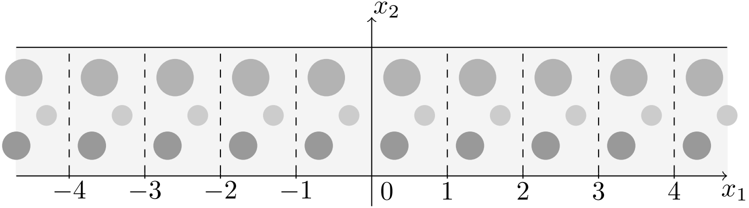
Helmholtz equation in wave guide

Wave-guide geometry: $\Omega := \mathbb{R} \times S$ with $S \subset \mathbb{R}^{d-1}$ bounded Lipschitz

Coefficient: $a : \Omega \to \mathbb{R}$ strictly positive, 1-periodic in x_1

Neumann condition on $\partial\Omega$



Given data: $f \in L^2(\Omega)$ with compact support, frequency $\omega \in \mathbb{R}$

Helmholtz equation

$$-\nabla \cdot (a\nabla u) = \omega^2 u + f \qquad \text{in } \Omega \tag{H}$$

Main result: existence of solutions

Important is the method: Only energy methods!

Literature

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Main result on periodic wave-guides

Flux conservation: For solutions φ of (H) and $-\infty < \rho < r < \infty$ holds

$$\operatorname{Im} \int_{\{\rho\}\times S} a\nabla\varphi \cdot e_1 \ \bar{\varphi} = \operatorname{Im} \int_{\{r\}\times S} a\nabla\varphi \cdot e_1 \ \bar{\varphi}$$

Solution concept: $u \in H^1_{\mathrm{loc}}(\Omega)$ with

(i) u solves (H) in Ω in the weak sense
(ii) $\sup_{r\in\mathbb{Z}} \|u\|_{L^2((r,r+1)\times S)} < \infty$
(iii) the radiation condition is satisfied

A loose formulation of (iii) on the right: There exist finitely many quasiperiodic homogeneous solutions φ_j of (H) with positive energy-flux such that $u - \sum_j \alpha_j\varphi_j \to 0$ as $x_1 \to \infty$.

Theorem (Existence and uniqueness result for periodic media)

Let data Ω, f, ω, and a be fixed, $a(x + e_1) = a(x)\ \forall x \in \Omega$. Let ω be a non-singular frequency (see below). Then there exists one and only one solution u to the radiation problem (i)–(iii).

Non-singular frequency

We use $Au := -\nabla \cdot (a \nabla u)$, cylinders $W_r := (r, r+1) \times S$, the norm $\|u\|_{sL} := \sup_{r \in \mathbb{Z}} \|u\|_{L^2(W_r)}$, and the space

$$X := \left\{ u|_{W_0} \,\middle|\, u \in H^1_{\mathrm{loc}}(\Omega),\ \|u\|_{sL} < \infty,\ Au = \omega^2 u \text{ in } \Omega \right\}$$

Definition (Non-singular frequency)

$\omega > 0$ is called a non-singular frequency (for coefficient a) if:

(a) Finite dimension: The space X has a basis $(\varphi_j)_{1 \le j \le M}$ with quasimoments $\xi_j \in [0, 2\pi)$ such that each φ_j possesses a ξ_j-quasiperiodic extension satisfying $A\varphi_j = \omega^2 \varphi_j$ in Ω.

(b) Non-vanishing flux: For every quasiperiodic function $u \in H^1_{\mathrm{loc}}(\Omega)$ with $Au = \omega^2 u$, the restriction $\varphi = u|_{W_0} \in X$ has the property

$$\mathrm{Im} \int_{W_0} a \nabla \varphi \cdot e_1 \bar{\varphi} \neq 0\,.$$

Exclusion of singular frequencies appears in all “blue” references

Right-going waves, projections, radiation condition

Construct basis $\phi_j^{\pm}$, flux of ϕ_j^{+}/ϕ_j^{-} is positive/negative
$u \in X$ can be written as $u = \sum_{j=1}^{N} \alpha_j \phi_j^{+} + \sum_{j=1}^{N} \beta_j \phi_j^{-}$
With, e.g., $\Pi_{X,+} : u \mapsto \sum_{j=1}^{N} \alpha_j \phi_j^{+}$ and orthogonal projection:

$$\Pi_{\pm} : L^2(W_0) \to X_{\pm} \subset L^2(W_0)$$

Π_{+}/Π_{-} projects onto right/left going base waves

Definition (Radiation condition)

Let ω be non-singular and $\Pi_{\pm}$ the above projections. We say that $u : \Omega \to \mathbb{C}$ with $\|u\|_{sL} < \infty$ satisfies the radiation condition if

$$\Pi_{-}(u|_{W_r}) \to 0 \quad \text{and} \quad \Pi_{+}(u|_{W_{-r}}) \to 0 \quad \text{as } r \to +\infty$$

We identify a function on W_r with a function on W_0 via a shift.

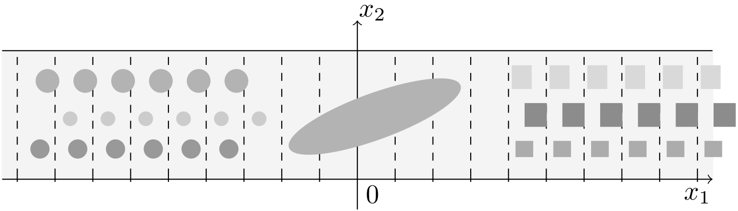
Non-periodic media

Theorem (Media that are periodic at infinity)

$\Omega = \mathbb{R} \times S$ be as above, $a : \Omega \to \mathbb{R}^{d\times d}$ essentially bounded, symmetric and positive. Periodicity outside a compact set: For some $R_0 > 0$

$$a(x + e_1) = a(x) \quad \text{for every } x \in \Omega \text{ with } |x_1| > R_0 \,,$$

$\omega > 0$ non-singular frequency for the left and *the right medium. If (i)–(iii) with $f = 0$ possesses only the trivial solution, then there is a unique solution u for arbitrary $f \in H^{-1}(\Omega)$ with compact support.*



The theorem has the character of a Fredholm alternative

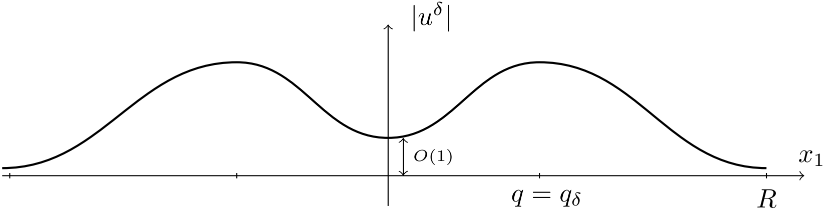
Failure 1: Try Limiting absorption

Solve on the infinite domain

$$-\nabla \cdot (a \nabla u^\delta) = (\omega^2 - i\delta) u^\delta + f$$

$\delta > 0$ provides decay at infinity

Normalized in central region, norms may look like this:



$\exists (q_\delta)_\delta : \quad |u^\delta(q_\delta)| \to \infty$

The limit $\delta \to 0$ cannot be performed!

On the proof

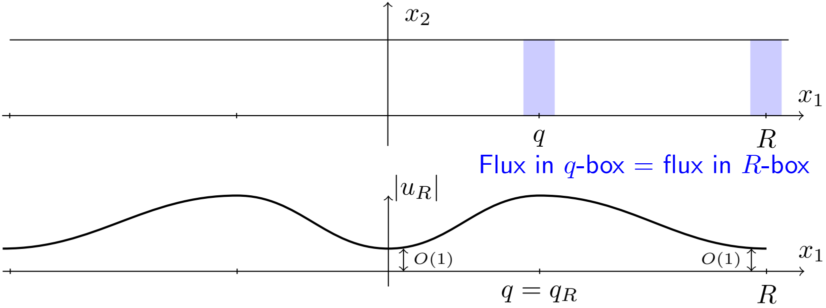
Good idea: Truncate to $\Omega_R := \Omega \cap \{-R-1 < x_1 < R+1\}$ and seek

$$u = u_R \in V_R := \left\{ u \in H^1(\Omega_R) \,\middle|\, u|_{W_R} \in X_+ \,,\ u|_{W_{-R-1}} \in X_- \right\}$$

which solves the Helmholtz equation between $-R$ and R.

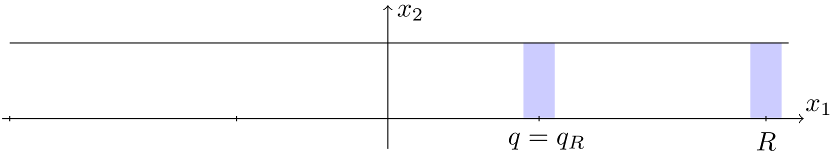
T. Dohnal and B. Schweizer. A Bloch wave numerical scheme for scattering problems in periodic wave-guides. SIAM J. Numer. Anal., 56(3):1848–1870, 2018.

Failure 2 (regarding uniform estimates)



u in q could be the sum of a large right-going and a large left-going wave!

The successful proof



- In W_q: $u = \Phi + \Psi + v$ with Φ right-going, Ψ left-going, v small
- Consider $U := u - \Phi$. Then: U is still right-going in W_R.
  And: U is approximately left-going in W_q
- Flux-equality for U implies that U is small in both boxes!
  - In W_R: $u \approx \Phi$
  - In W_q: $\Psi \approx 0$

  Hence: u is similar (in norms) in W_R and W_q
- Contradiction argument, distinguishing cases regarding the positions $q = q_R$ of "bad" boxes: Solution sequence remains bounded

Conclusions

We have shown: **existence result with energy methods**

- periodic media: existence and uniqueness
- piecewise periodic media: Fredholm alternative

Open questions

- limiting absorption principle
- characterization of non-singular frequencies
- beyond wave-guide geometries

Thank you!