CORRECTIONS and UPDATES concerning

Inhomogeneous Helmholtz equations in wave-guides – existence and uniqueness results with energy methods

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We have to make comments on the proofs in the article [5]: B. Schweizer, Inhomogeneous Helmholtz equations in wave guides – existence and uniqueness results with energy methods, European J. Appl. Math. 34 (2023), no. 2, 211–237. In that article, we obtain existence results for the Helmholtz equation

$$-\nabla \cdot (a \nabla u) - \omega^2 u = f \tag{0.1}$$

in an unbounded wave-guide $\Omega := \mathbb{R} \times S \subset \mathbb{R}^d$, where $S \subset \mathbb{R}^{d-1}$ a bounded domain and a is a strictly elliptic coefficient function. Theorem 1.1 of [5] treats the case that a is periodic in the unbounded direction $x_1 \in \mathbb{R}$. Theorem 1.2 treats the case that, for some $R_0 > 0$, the coefficient a coincides with a periodic function a^{right} for $x_1 > R_0$ and with another periodic function a^{left} for $x_1 < -R_0$.

1. Problems regarding concentrations near artificial boundaries

We believe that Theorems 1.1 and 1.2 are correct as stated in [5]. Unfortunately, there is a (solvable) problem in the proofs. We want to explain here the problem and possible solutions. A simple solution is to introduce an additional assumption; this would require only minimal changes in the text (but modify the statements of Theorems 1.1 and 1.2). The other solution is to modify the proof to a larger extend.

The flaw in the argument occurs in the proofs of Lemmas 4.6 and 5.1. In both proofs, in Case 2b, "Large values near boundaries", we derive, in the last paragraph, a contradiction using the odd extension of the solution. This argument can only be used when the medium is x_1-symmetric, $a(x_1, \tilde{x}) = a(-x_1, \tilde{x})$ for all $x_1 > 0$ and all $\tilde{x} \in S$. Accordingly, for Theorem 1.2, the argument works when both media, a^{left} and a^{right} are x_1-symmetric. This means that the proof of the theorems is correct for symmetric media; but the symmetry assumption is not formulated in the theorems.

In the following, we explain how the proofs must be corrected in order to have a proof of the theorems as stated.

1.1. Edge resonances.

The proofs can be adapted to non-symmetric media with only minimal changes, when an additional assumption on ω is introduced in the theorems. Before we outline the correct proof of the theorems (in the next subsection), we explain how one can work with an additional assumption. We give the explanation only for Theorem 1.1.

One can use the additional assumption that ω^2 is not contained in the set of edge-resonances $\omega^2 \notin \sigma_{\text{edge}} := \sigma^+_{\text{edge}} \cup \sigma^-_{\text{edge}}$. Let us include the definition of this

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condition. There holds $\omega^2 \in \sigma^-_{\text{edge}} = \sigma^-_{\text{edge}}[a]$ if, and only if, in the half wave-guide $\Omega^- := \mathbb{R}_- \times S$, there is a non-trivial solution $u^- \in H^1(\Omega^-)$ of the problem

$$\begin{aligned} -\nabla \cdot (a \nabla u^-) - \omega^2 u^- &= 0 \quad \text{in } \Omega^-\,, \\ u^- &= 0 \quad \text{on } \partial\Omega^-\,. \end{aligned} \tag{1.1}$$

The definition of σ^+_{edge} is the corresponding condition using $\Omega_+ := \mathbb{R}_+ \times S$. For further information we refer to [1], a similar condition was also used in [2]. For numerical results concerning the existence of edge-resonances see, e.g., [4].

Under the additional assumption $\omega^2 \notin \sigma_{\text{edge}} = \sigma_{\text{edge}}[a]$, the proof of Theorem 1.1 is correct. It is sufficient to replace the last three sentences of the proof of Lemma 5.1 by the observation that the limit function contradicts the new assumption.

For this modified proof, it is worth mentioning why the limit function $\tilde{v}$ is of class $H^1(\Omega^-)$. One can multiply the function $\tilde{v}$ with a cut-off function θ that satisfies $\theta(x) \equiv 1$ for $x_1 < -1$, the new function satisfies an inhomogeneous Helmholtz equation on the domain $\mathbb{R} \times S$ and satisfies a radiation condition on the left. It is therefore of class $H^1(\mathbb{R} \times S)$ by classical results that can found, e.g., in [3].

Accordingly, for Theorem 1.2, one has to make the additional assumption $\omega^2 \notin \sigma^-_{\text{edge}}[a^{\text{right}}] \cup \sigma^+_{\text{edge}}[a^{\text{left}}]$.

Regarding the broader perspective of this point: For symmetric media, the set of edge-resonances is empty, $\sigma_{\text{edge}} = \emptyset$. This can be shown by the above-mentioned odd-extension argument, for a discussion see [1].

1.2. **Treatment of edge resonances.** As stated above, the theorems are correct even without an assumption on edge resonances. This requires more modifications of the proof. We sketch the necessary changes.

We denote by E^- the space of homogeneous solutions u^- to (1.1) with the additional requirement that the trivial extension of u^- to the right is a solution of the truncated problem for $f = 0$ (with the right boundary in $x_1 = 0$).

In the proofs of the lemmas one must modify the constructed solution sequence. We demonstrate this with Lemma 5.1. The proof starts with the text "We use a sequence of solutions u_k to the truncated problems". This must be replaced as follows: One starts with the solutions $\tilde{u}_k$ to the truncated problems. Then, one constructs u_k by subtracting from $\tilde{u}_k$ appropriate elements of E^- (more precisely: elements of E^- multiplied with cut-off functions in order not to interfere with the left boundary). With this modification, one achieves that: (i) u_k is an approximate solution of the truncated problem and (ii) any limit of $u_k(\cdot - R_k e_1)$ is orthogonal to E^-. With this new construction of an approximate solution sequence u_k, our proof works.

2. On a simplification of the non-degeneracy assumption

In [5], we assumed that ω is non-singular in the sense of Definition 2.3. We want to comment two points: 1. The assumption can (and should) be weakened — without changes in the further text. 2. Due to new results, we know that part (a) of the non-singularity assumption is automatically satisfied.

Before we describe the simplifications, we recall that the article introduced in (2.1) the two forms Q and Q^s (corresponding to the medium a, where $l > 0$ is the periodicity of that medium in x_1-direction and $W_0 := (0, l) \times S$ is a periodicity cell):

$$Q(u,v) := \frac{1}{l} \int_{W_0} a \nabla u \cdot e_1 \bar{v} \quad \text{and} \quad Q^s(u,v) := \frac{1}{2} \left(Q(u,v) - \overline{Q(v,u)} \right). \tag{2.1}$$

Theorems 1.1 and 1.2 of [5] impose an assumption on ω. The frequency ω should be non-singular in the sense of Definition 2.3, which demands a finite dimensionality condition in (a) and a non-vanishing flux condition in (b).

2.1. **A weaker assumption.** Part (b) in Definition 2.3 demands $\operatorname{Im} Q(\varphi,\varphi) \neq 0$ for all φ in X. This is not necessary for the subsequent proof. It is sufficient to demand that $Q^s(\varphi,.)$ is a non-trivial form for every $\varphi \neq 0$. This implies that there is a basis such that every basis function is either a right-going or a left-going wave (as in the discussion after Definition 2.3).

2.2. **Simplification due to new results.** For every quasimoment $\alpha \in \mathbb{R}$, let Y^α be the space of α-quasiperiodic solutions ϕ to the homogeneous Helmholtz equation (for the given frequency ω and the periodic medium a). Assume (as indicated above) that, for every $0 \neq \varphi \in Y^\alpha$, the map $Q^s(\varphi,\cdot) : Y^\alpha \to \mathbb{C}$ is non-trivial. Under these conditions, the new results of [3] imply that ω satisfies all conditions that are necessary for Theorem 1.1 of [5]. Regarding Theorem 1.2, one has to assume the above non-triviality for both media a^{left} and a^{right}.

Loosely speaking, the new results of [3] imply the following: One keeps condition (b) on the non-vanishing flux (in modified form), but the old condition (a) is not necessary. Let us emphasize that, technically, we worked with the equation $-\Delta u - k^2 n u = f$ in [3], but the methods remain applicable for the case considered here.

Let us comment on the proof of the above implication. We note that E of (3.10) in [3] coincides, up to pre-factors, with Q^s. Assumption 3.5 of [3] is the above condition that $Q^s(\varphi,\cdot)$ is non-trivial. Theorem 6.2 of [3] yields a characterization of the space X of Definition 2.3 of [5].

3. Minor remarks

In Proposition 4.5: The choice of subsequences is intricate in Proposition 4.5. The following typo is therefore very confusing: In Step 2 (a), $R_k - r_k$ appears. There should not appear r_k, but only r_m. The rest of the proof is ok, it works with $r = r_m$, $k = k_m$, $R = R_k = R_{k_m}$.

In Lemma 5.1: In the proof of Lemma 5.1, Case 2a, "Interior points", last paragraph, we have written: "exchanging Π_+ with Π_-, we find the same estimate for $\Pi_+(v_k|_{W_\rho})$." The argument must be given slightly different, since subtracting left-going waves is not allowed in the energy-flux equality.

One has to argue as follows, using (5.2), which provides that $v_k|_{W_{R_k}}$ is small: The total energy-flux of v_k is small in the radiation box W_{R_k}, hence the total energy-flux of v_k is also small in W_ρ. Since the left-going part of $v_k|_{W_\rho}$ is small (already shown in (5.4)), also the right-going part must be small.

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