(4)

$D = B_r(0) \subset \mathbb{R}^2$

$$\int_D \nabla \cdot v = 0 \quad (??)$$

$$\int_{\partial D} v \cdot \vec{n} = \int_{\partial D} \frac{1}{r} \vec{n} \, \vec{n} = 2\pi r \cdot \frac{1}{r}$$

$$= 2\pi$$

Die Divergenz ist in 0 nicht definiert

$$\rightsquigarrow \quad \nabla \cdot v = 2\pi \delta_0$$

Im Distributionssinn (Dirac)

Was ist v ?

$$\phi(x) = \log |x|$$

$$\Rightarrow \nabla \phi(x) = \frac{1}{2} \frac{1}{|x|^2} 2x = \frac{x}{|x|^2}$$

$$\frac{1}{2} \log |x|^2$$

$$\nabla |x|^2 = 2x$$

2023-10-23

Weitere Anwendungen Partielle Integration, ..

$$v := \vec{f}\, g \qquad \vec{f}: D \to \mathbb{R}^n \qquad g: D \to \mathbb{R}$$

$$\int_{\partial D} g \vec{f}\, \vec{n} = \int_{\partial D} v \cdot \vec{n} = \int_D \nabla \cdot v = \int_D (\nabla \vec{f}) g + \vec{f} \cdot \nabla g$$

$$\nabla \cdot v = \sum_{j=1}^{n} \partial_j v_j = \sum_j \partial_j (\vec{f}_j g) = \sum_j \partial_j \vec{f}_j\, g + \vec{f}_j\, \partial_j g$$

Falls zum Beispiel $g|_{\partial D} = 0$

$$\int_D g \nabla \cdot \vec{f} = -\int_D \nabla g\, \vec{f}$$

$$\int_D \partial_1 g \; - \; ?$$

$g: D \to \mathbb{R}$

1

$$\int_D \nabla\cdot(g e_1) = \int_{\partial D} g e_1 \vec{n}$$

Part. Int.

$f, g: D \to \mathbb{R}$

$(g|_{\partial D} = 0)$

$$\int_D g\,\partial_1 f = -\int_D \partial_1 g\, f$$

Randintegrale über Normalenvektoren

$v = e_1, \quad \nabla\cdot v = 0$

$$0 = \int_D \nabla\cdot v = \int_{\partial D} \vec{n}\, v = \int_{\partial D} \vec{n}\, e_1$$

Also $\int_{\partial D} \vec{n} = 0$

Bsp: $v = f \times g$

$v, f, g : \mathbb{R}^3 \to \mathbb{R}^3$

$\nabla \cdot v = \nabla \times f \cdot g - f \cdot \nabla \times g$

Gauß:

$$\int_D \nabla \cdot (f \times g) = \int_{\partial D} (f \times g) \cdot \vec{n}$$

$$\| $$

$$\int_D \nabla \times f \cdot g - f \cdot \nabla \times g$$

$\rightsquigarrow$ Form der Part. Int. $\int \overset{\frown}{\operatorname{rot} f \cdot g} = \int f \cdot \operatorname{rot} g$

Für $g \in \mathbb{R}^3$ (eine konst. Fkt.)

$$g \cdot \int_D \nabla \times f = -g \cdot \int_{\partial D} f \times \vec{n}$$

$a \times b \cdot c$
$= -b \cdot (a \times c)$

$g \in \mathbb{R}^3$ bel.

Also:

$$\int_D \nabla \times f = -\int_{\partial D} f \times \vec{n}$$

(falls $\int_{\partial} = 0$)

2023-10-23

$f: \partial D \to \mathbb{R}^n$

$$\int_{\partial D} f \in \mathbb{R}^n \text{ definiert durch } e_i \cdot \int_{\partial D} f = \int_{\partial D} f\, e_i \quad \forall_{i \le n}$$

Falls $f, g: D \to \mathbb{R}$ glatt genug

$$\int_D \Delta f \cdot g = \int_D (\nabla \cdot \nabla f)\, g = -\int_D \nabla f\, \nabla g + \int_{\partial D} \vec{n}\, \nabla f\, g$$

$$= +\int_D f\, \Delta g - \int_{\partial D} f\, \nabla g\, \vec{n} + \int_{\partial D} g\, \nabla f\, \vec{n}$$

Formel

$$\int_D \Delta f\, g - \int_D f\, \Delta g = \int_{\partial D} (g \nabla f - f \nabla g) \cdot \vec{n}$$

"Integralformel von Green"

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Rechnungen in der Ebene

$n = 2, \quad \mathbb{R}^2 \supset D$

$u : D \to \mathbb{R}$

$v : D \to \mathbb{R}^2$

$\nabla u(x) = \begin{pmatrix} \partial_1 u \\ \partial_2 u \end{pmatrix}(x) \qquad \nabla^\perp u(x) = \begin{pmatrix} -\partial_2 u \\ \partial_1 u \end{pmatrix}$

$\nabla \cdot v(x) = (\partial_1 v_1 + \partial_2 v_2)(x)$

$\nabla^\perp \cdot v(x) = (-\partial_2 v_1 + \partial_1 v_2)(x)$

$\nabla^\perp$ ist wie dritte Komponente von rot $\begin{pmatrix} v_1 \\ v_2 \\ 0 \end{pmatrix}$

$v^\perp(x) = \begin{pmatrix} -v_2 \\ v_1 \end{pmatrix}(x)$

$\nabla^\perp v = -\partial_2 v_1 + \partial_1 v_2 = -\nabla \cdot (v^\perp)$

Rechne mit Gauß

$$\int_D \nabla^\perp v = -\int_D \nabla \cdot (v^\perp) \overset{(G)}{=} -\int_{\partial D} \vec{n} \cdot v^\perp$$

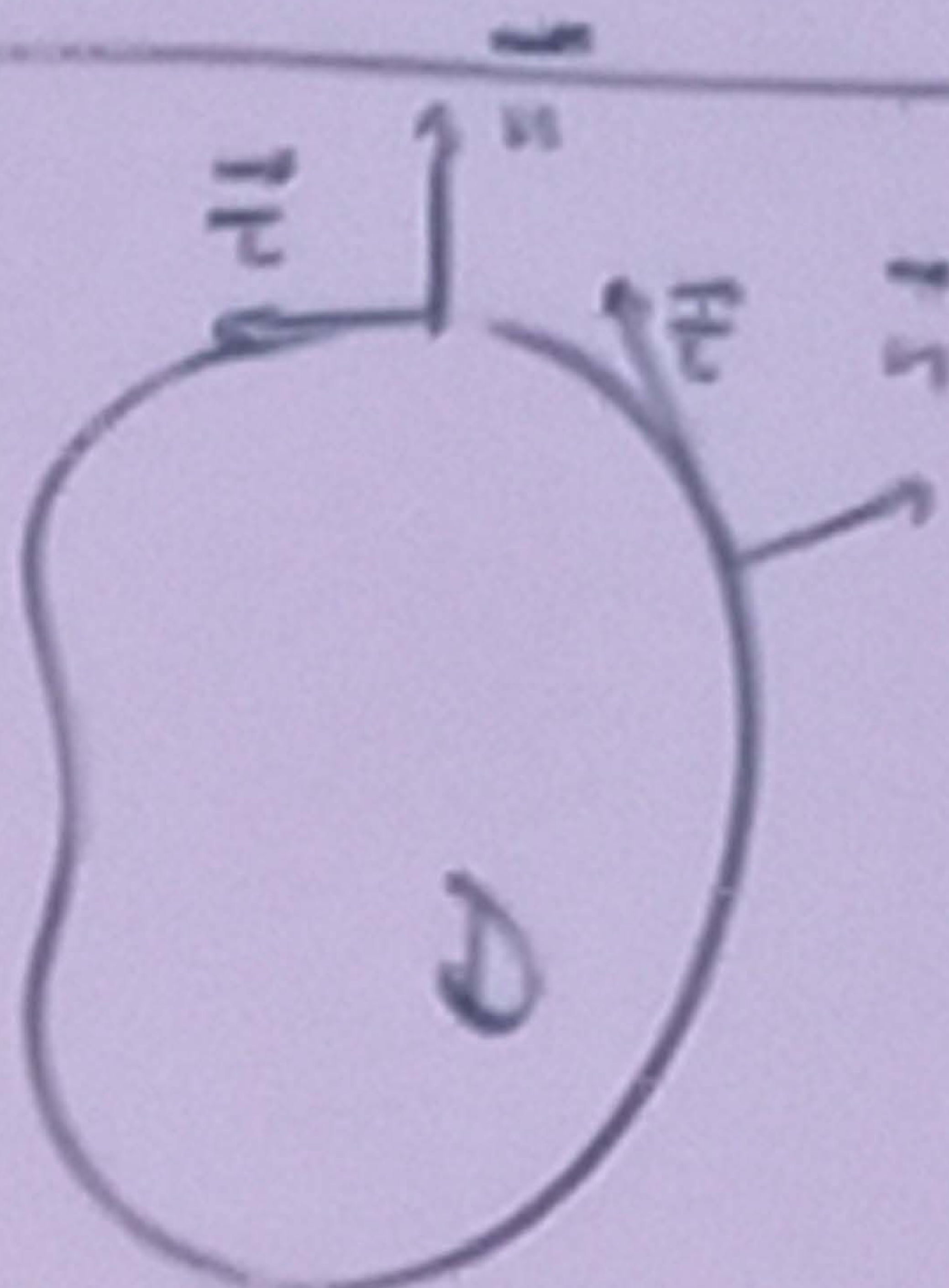
$$= \int_{\partial D} v \cdot n^\perp$$

2023-10-23

$$\begin{pmatrix}a_1\\a_2\end{pmatrix}\begin{pmatrix}b_1\\b_2\end{pmatrix}^\perp = \begin{pmatrix}a_1\\a_2\end{pmatrix}\begin{pmatrix}-b_2\\b_1\end{pmatrix}$$

$$= -a_1 b_2 + a_2 b_1$$

$$\begin{pmatrix}-a_2\\a_1\end{pmatrix}\begin{pmatrix}b_1\\b_2\end{pmatrix} = -\left(\qquad \% \qquad \right)$$



$$\vec{\tau} = \vec{n}^\perp$$

$$\int_D \nabla^\perp \cdot v = \int_{\partial D} v \cdot \vec{\tau}$$

"Satz von Green"

"Satz von Stokes in 2D"

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Gauß

$$\int_D \nabla \cdot v = \int_{\partial D} v \cdot \vec{n}$$

Folgerungen: Setze

- $v = f e_j \rightsquigarrow \int \partial_j f = \int_{\partial} f \vec{n}_j$
- $v = \vec{f} g \rightsquigarrow$ Part. Int.
- $v = e_j \times \vec{f}$

l.S.: $\int_D \operatorname{rot} \vec{f}$

2 Raumdim.:

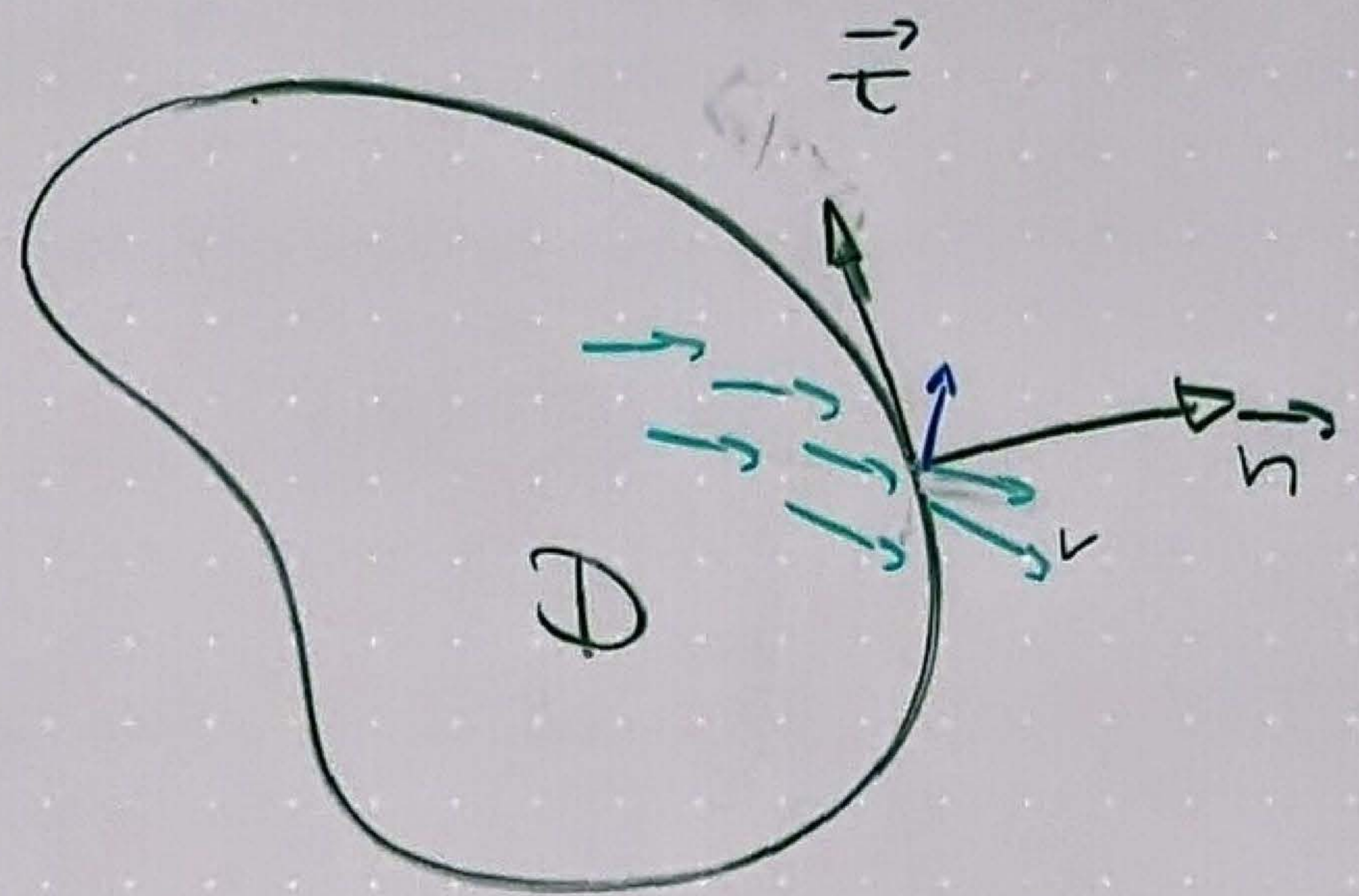
$v = (v_1, v_2)$

$\nabla \cdot v = \partial_1 v_1 + \partial_2 v_2$

Satz von Green

$$\int_{D} \left(-\partial_2 v_1 + \partial_1 v_2\right) = \int_{D} \nabla \cdot \begin{pmatrix} v_2 \\ -v_1 \end{pmatrix} = \int_{\partial D} \vec{n} \cdot \begin{pmatrix} v_2 \\ -v_1 \end{pmatrix}$$

$$= \int_{\partial D} v \cdot \vec{\tau}$$



$n = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\tau = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\begin{pmatrix} v_2 \\ -v_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

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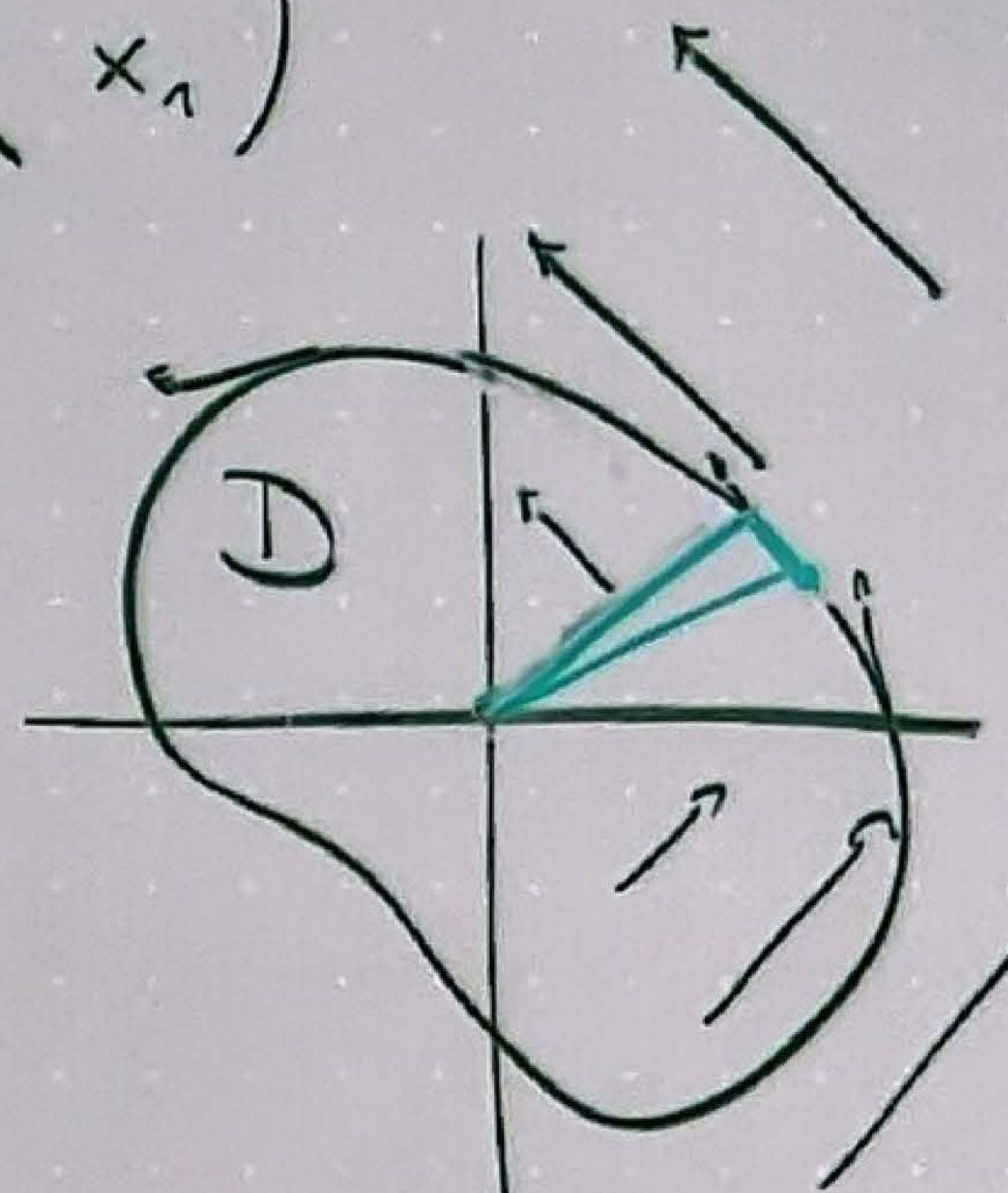
Beispiel

$$v(x) = (v_1, v_2)(x_1, x_2) = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}$$



$$-\partial_2 v_1 + \partial_1 v_2 = 1 + 1 = 2$$

$D \subset \mathbb{R}^2$

$0 \in D$

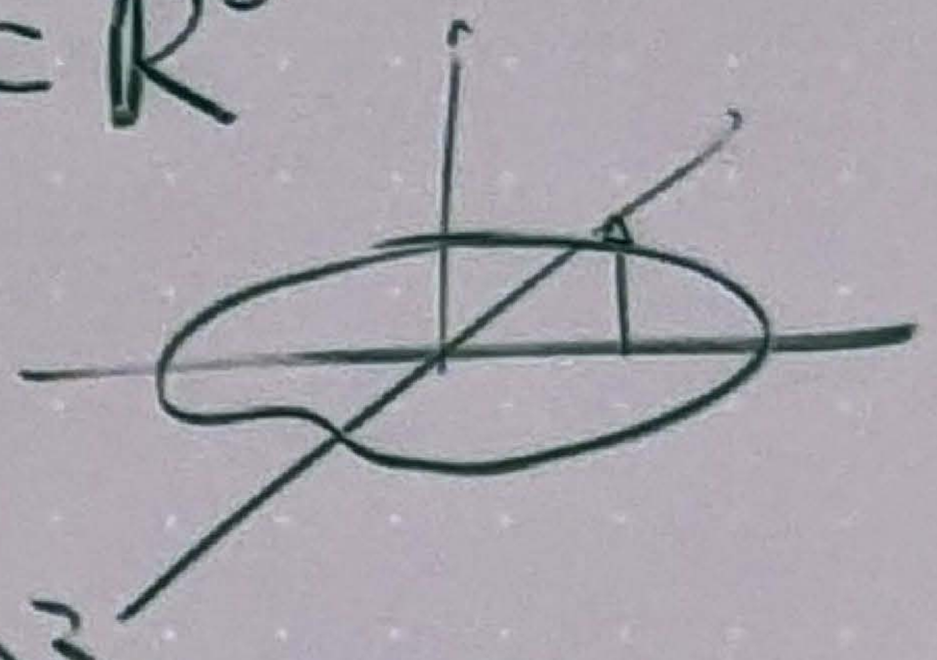
Green: $\displaystyle\int_{\partial D} v \cdot \tau = \int_{\partial D} \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} \cdot \tau = 2 \int_{D} 1 = 2\, \mathrm{Vol}_2(D)$

Volumenberechnung!

Satz von Stokes

Zunächst: $D \subset \mathbb{R}^2$

$$\Gamma = D \times \{0\} \subset \mathbb{R}^3$$



Vektorfeld

$$w : \mathbb{R}^3 \to \mathbb{R}^3$$

$$w(x_1, x_2, x_3) = w(x_1, x_2) = \begin{pmatrix} w_1(x_1, x_2) \\ w_2(x_1, x_2) \\ 0 \end{pmatrix}$$

$$\operatorname{rot} w(x) = \begin{pmatrix} \partial_2 w_3 - \partial_3 w_2 \\ \partial_3 w_1 - \partial_1 w_3 \\ \partial_1 w_2 - \partial_2 w_1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ \partial_1 w_2 - \partial_2 w_1 \end{pmatrix}$$

$\vec{n}$ auf Γ: $\quad \vec{n} = e_3$

$$(\operatorname{rot} w) \cdot e_3 = \partial_1 w_2 - \partial_2 w_1$$

$$\boxed{\int_\Gamma (\operatorname{rot} w) \cdot \vec{n} = \int_{\partial\Gamma} w \cdot \vec{\tau}}$$

2023-10-25

Also: Satz von Green ist der Satz von Stokes für "flache Flächen"

Allgemeiner Stokes

Sei $D \subset \mathbb{R}^2$ offen, C^1-Rand

Sei $\phi : D \to \mathbb{R}^3$ injektiv, C^1

$$\Gamma := \phi(D)$$

$v : \mathbb{R}^3 \to \mathbb{R}^3$ sei C^1

Dann gilt

$$\boxed{\int_\Gamma (\operatorname{rot} v) \cdot \vec{n} = \int_{\partial^* \Gamma} v \cdot \vec{\tau}}$$

wobei: $\vec{n}$ ein Normalen-Vektorfeld auf Γ, stetig

$\vec{\tau}$ Tangentialen-Feld auf

$$\partial^* \Gamma := \phi(\partial D)$$

(Kurve im $\mathbb{R}^3$)

$(\partial\Gamma \neq \overline{\Gamma})$

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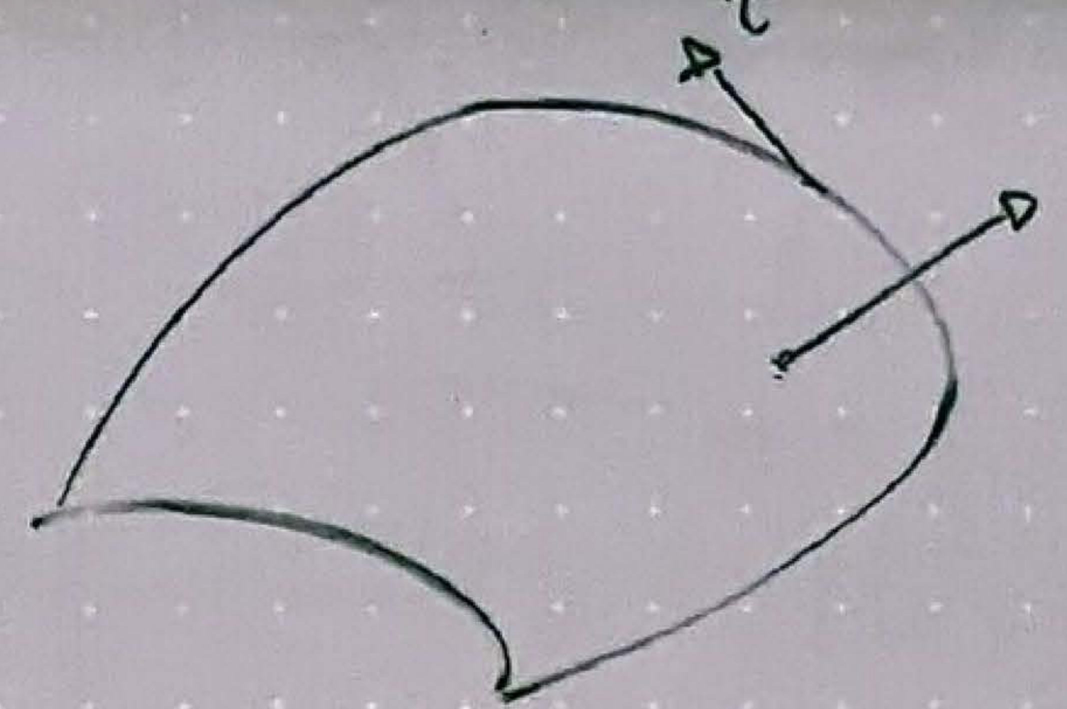
Orientierung muss richtig sein!



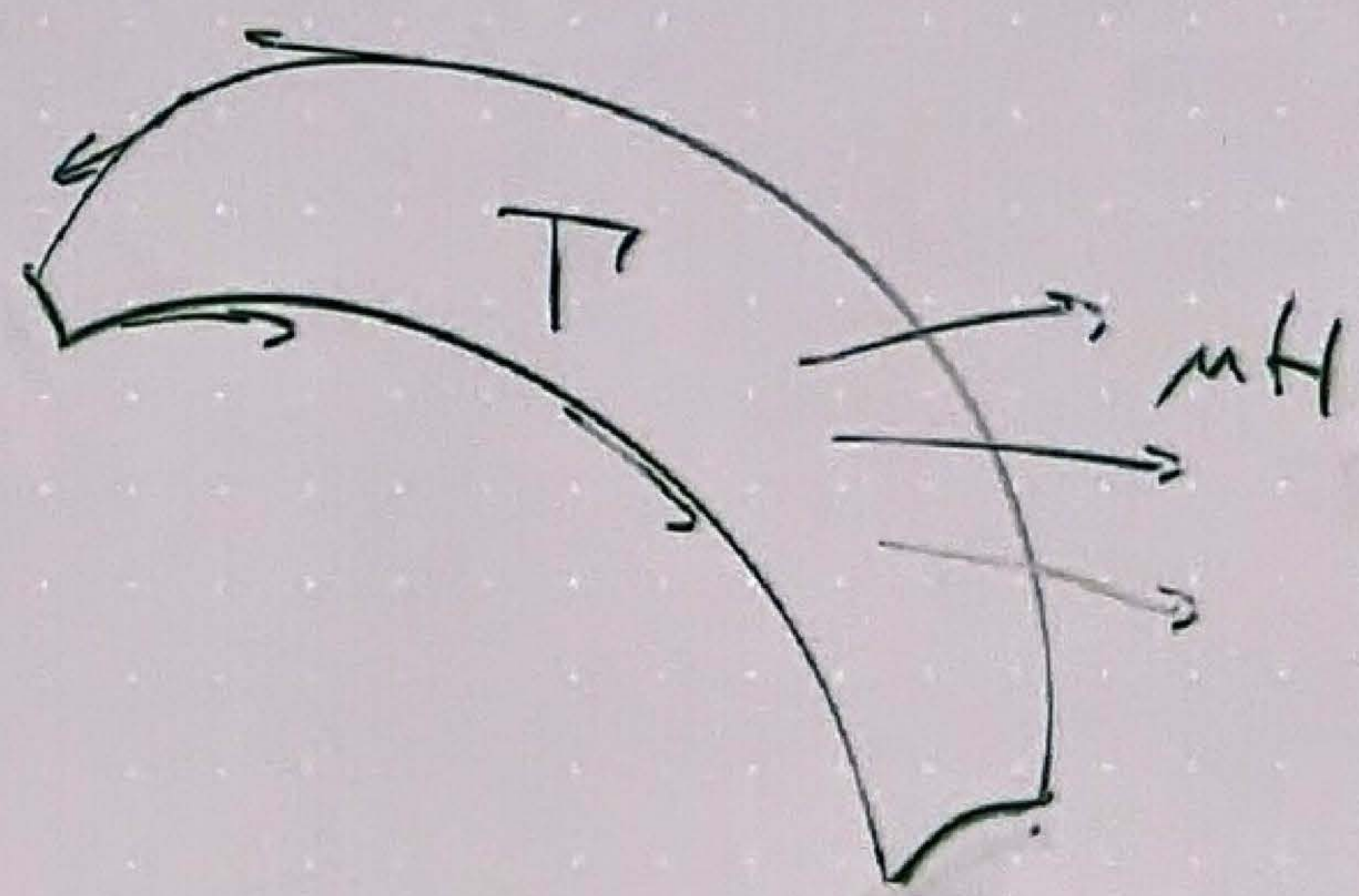
„$\vec{n}$ nach oben, dann
$\vec{\tau}$ so, dass die Randkurve math. pos. durchlaufen wird".

2023-10-25

Ausblick: Maxwell-Gleichung

Zeitharmonisch: $e^{-i\omega t}$

$$\boxed{\begin{aligned} \operatorname{rot} E &= i\omega\mu H \\ \operatorname{rot} H &= -i\omega\varepsilon E \end{aligned}}$$



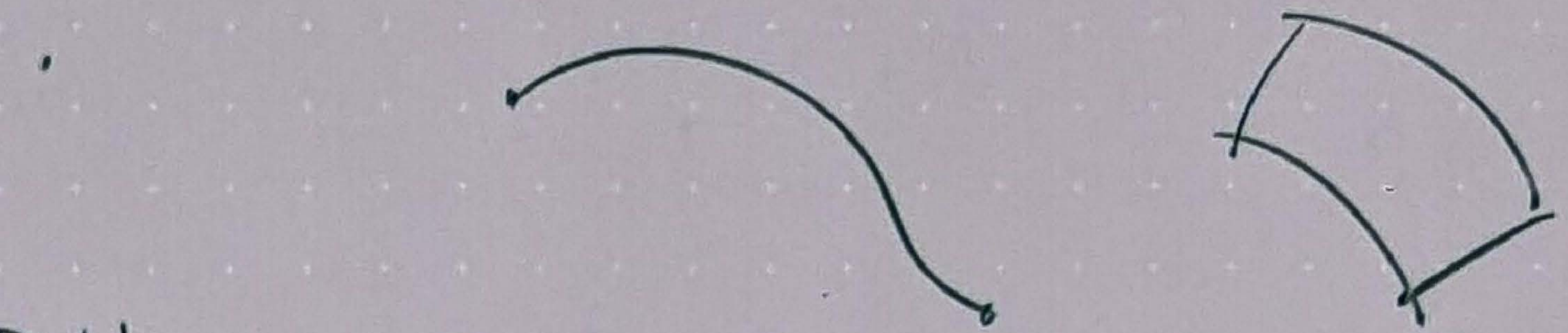
magn. Fluss durch Γ

$$i\omega \int_{\Gamma} \mu H \cdot \vec{n} \overset{!}{=} \int_{\partial\Gamma} E \cdot \vec{\tau}$$

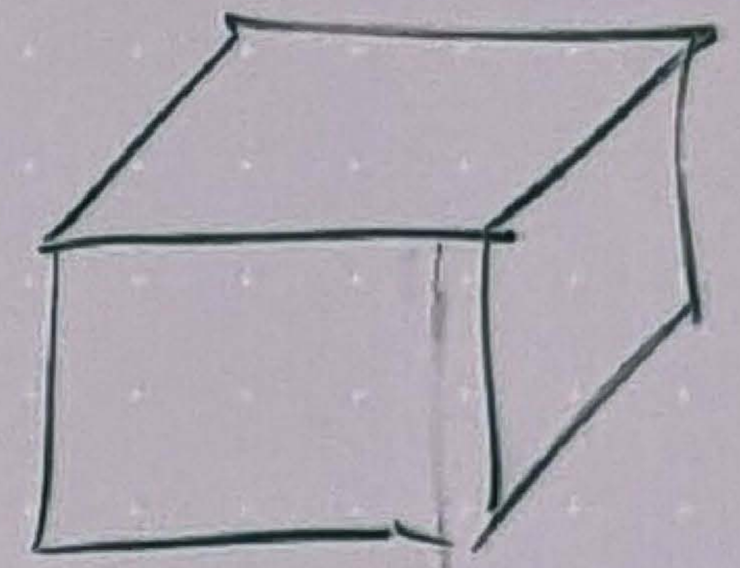
"Höhere" Perspektive:

Differentialformen

$u \longmapsto \nabla u = v \longmapsto \mathrm{rot}\, v = w \longmapsto \mathrm{div}\, w = f$



Punkte Linien Flächen



Volumina

Allg. Stokes

$$\int_M d\lambda = \int_{\partial M} \lambda$$

$$\int_\Gamma \nabla u \cdot \tau = u(\text{Endpt.}) - u(\text{Anfpt.})$$

M Fläche $\Rightarrow d\lambda = \mathrm{rot}$

Stokes

$$\int_\Gamma \mathrm{rot}\, v = \int_{\partial \Gamma} v \cdot \tau$$

M Volumen $\Rightarrow d\lambda = \mathrm{div}$

Gauß

$$\int_M \mathrm{div}\, w = \int_{\partial M} w \cdot \vec{n}$$

M Linie $\Rightarrow d\lambda = \nabla$

$$\int_M \nabla u \cdot \tau = \int_{\partial M} u$$

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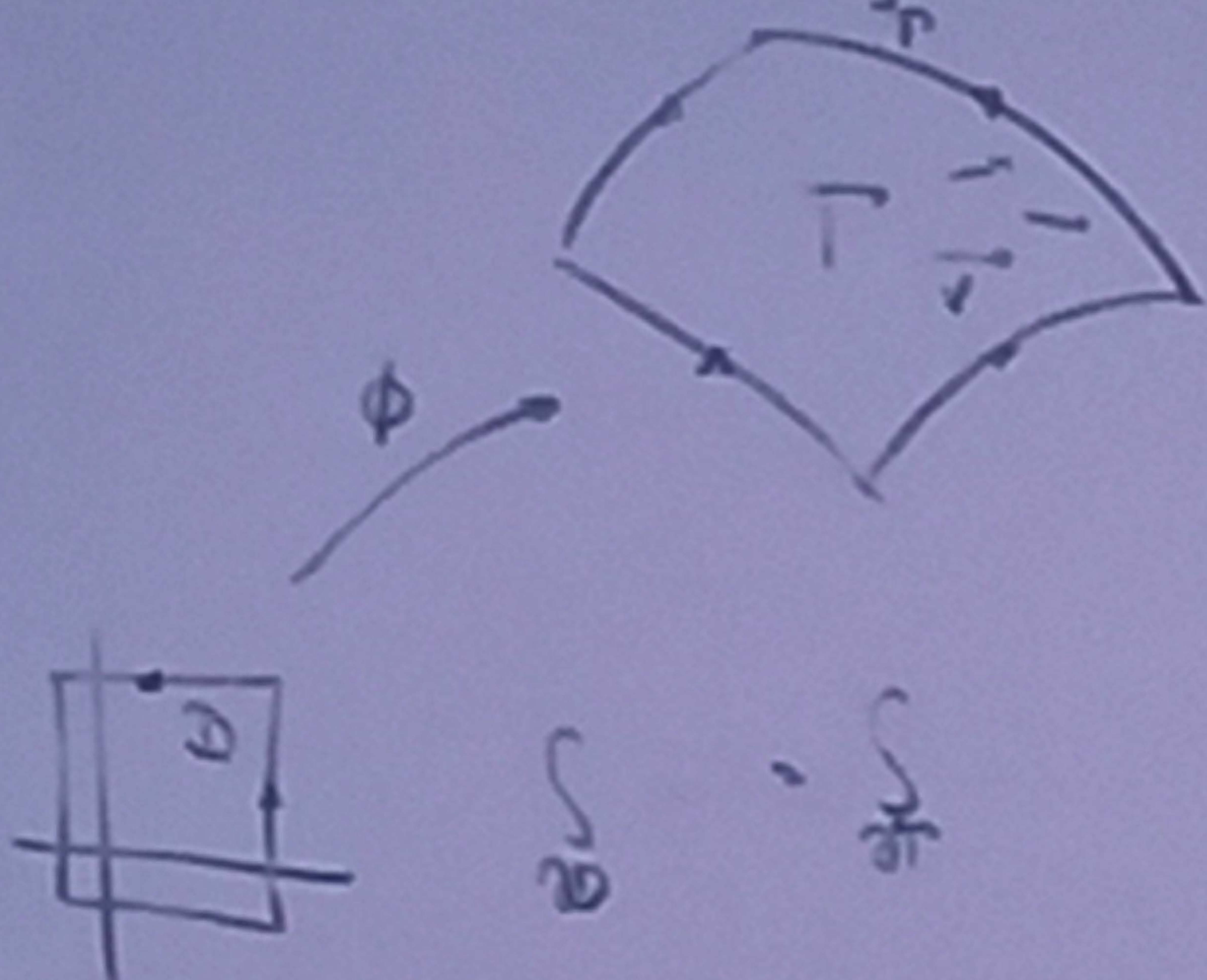
Beweis von Stokes

$\int \operatorname{rot} v \cdot \vec n = ?$



$\int_{\partial D} - \int_{\partial \Gamma}$