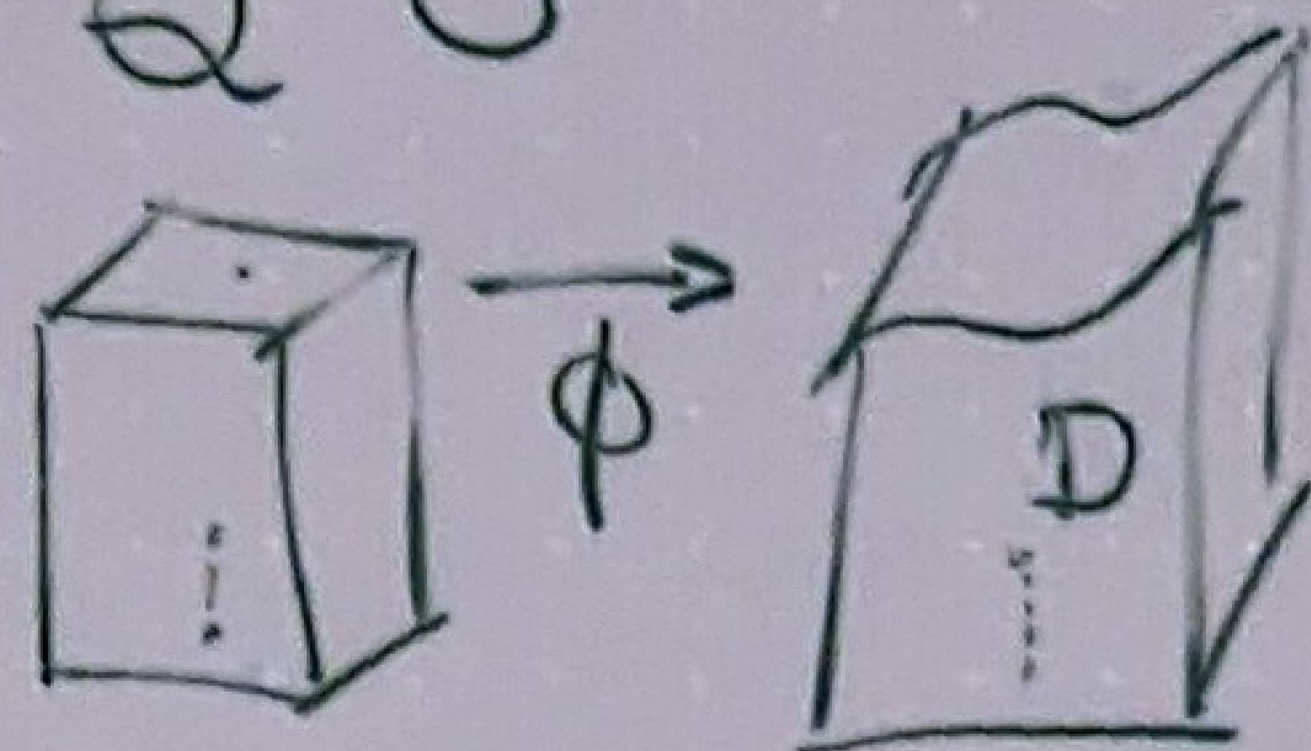


Volumenintegral

$$\int_D f = \int_Q \int_0^1 f \circ \phi \boxed{1}$$



$$(\tilde{x}, x_n) \xrightarrow{\phi} (\tilde{x}, h(\tilde{x}) + x_n - 1)$$

Stellen uns vor: $h(\tilde{x}) \approx 1$

v verschwindet in Umgebung der Unterseite

$$\det D\phi^T \cdot D\phi = 1$$

$$\text{Also } \sqrt{g} = 1$$

Rechne Gauß nach:

$$1. \text{ Fall: } v = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ v_n \end{pmatrix}$$

$$\int_{\partial} \nabla \cdot \mathbf{v} = \int_{\partial} \partial_n v_n = \int_Q \int_0^{h(\tilde{x})} \partial_n v_n dx_n d\tilde{x} = \int_Q v_n \Big|_0^{h(\tilde{x})} d\tilde{x}$$

$$\int_{\partial D} \mathbf{v} \cdot \vec{n} = \int_{\Gamma} \mathbf{v} \cdot \vec{n} = \int_Q \underbrace{\frac{v_n}{\sqrt{1 + \sum_j |\partial_j h|^2}}}_{\text{①}} \underbrace{\sqrt{\quad}}_{\text{②}} d\tilde{x}$$

$$= \int_Q v_n(\tilde{x}, h(\tilde{x})) d\tilde{x}$$

$$= \int_Q v_n(\tilde{x}, h(\tilde{x})) d\tilde{x}$$

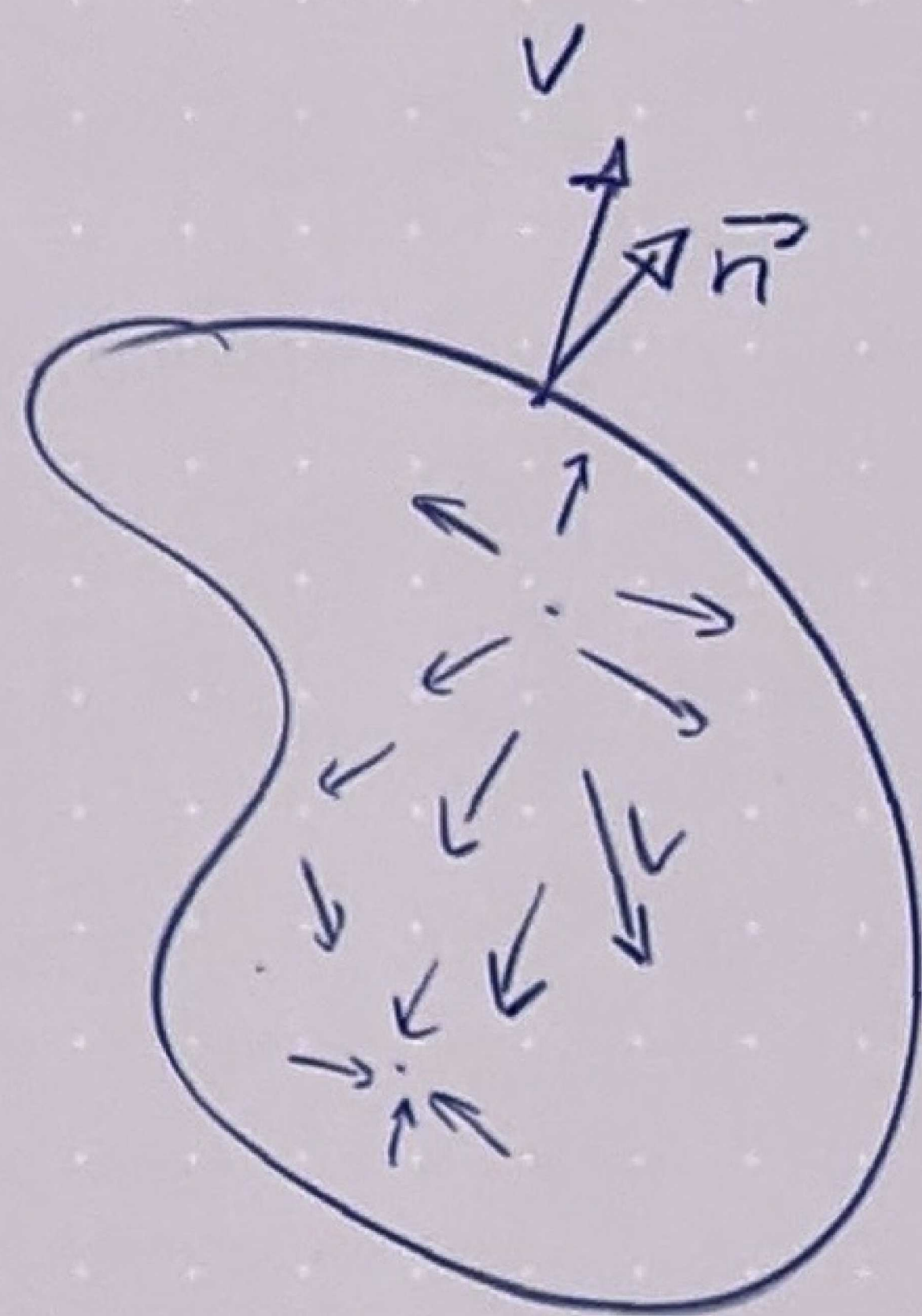


Also $\int_{\partial} \nabla \cdot \mathbf{v} = \int_{\partial D} \mathbf{v} \cdot \vec{n}$

4

Ziel:

$$\int_D \nabla \cdot v = \int_{\partial D} v \cdot \vec{n}$$



Quellen: $\nabla \cdot v$
 $= \operatorname{div}(v)$

$(v \cdot n)(x)$ Ausfluss
 im Punkt $x \in \partial D$

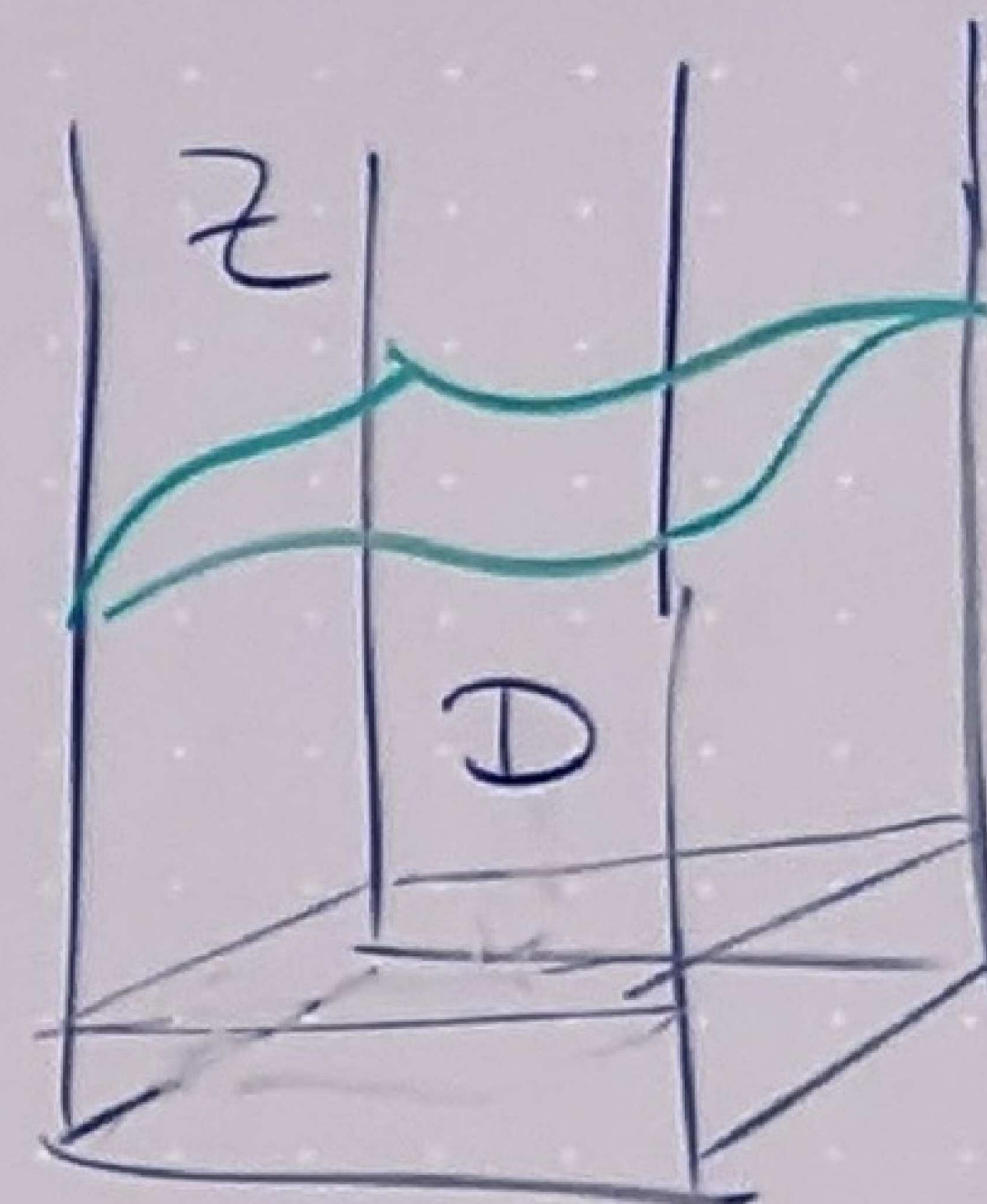
$$Q = (0, 1)^{n-1}$$

①

$$Z = Q \times \mathbb{R}_+$$

$$h: Q \rightarrow \mathbb{R}$$

diffbar, strikt positiv



$$D := \{(\tilde{x}, x_n) \mid \tilde{x} \in Q, 0 < x_n < h(\tilde{x})\}$$

$$v: D \rightarrow \mathbb{R}^n \text{ diffbar}$$

v verschw. auf ∂Z , sogar:
 In Umgebung davon

$$T := \{ (\tilde{x}, h(\tilde{x})) \mid \tilde{x} \in Q \}$$

Vorgehen: Berechne $\int_D \nabla \cdot v$

Dafür:

$$\vec{n}(x) = \frac{(-\partial_1 h(\tilde{x}), \dots, -\partial_{n-1} h(\tilde{x}), 1)}{\sqrt{1 + \sum_{j=1}^{n-1} |\partial_j h(\tilde{x})|^2}}$$

$$\int_T f = \int_Q f(\tilde{x}, h(\tilde{x})) \sqrt{1 + \sum_{j=1}^{n-1} |\partial_j h(\tilde{x})|^2} d\tilde{x}$$

" $\sqrt{g} = \sqrt{\det D\phi^T \cdot D\phi}$ "

$$f: T \rightarrow \mathbb{R}$$

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1. Fall: $v = (0, \dots, 0, v_n)$

②

$$\int_D \nabla \cdot v = \int_D \partial_n v_n = \int_Q \int_0^{h(\tilde{x})} \partial_n v_n(\tilde{x}, x_n) dx_n d\tilde{x}$$

Hauptsatz $\rightarrow$ $\int_Q v_n(\tilde{x}, h(\tilde{x})) d\tilde{x}$

Zum Vergleich:

$$\int_T v \cdot \vec{n} = \int_Q v(\tilde{x}, h(\tilde{x})) \cdot \frac{(-\partial_1 h, \dots, -\partial_{n-1} h, 1)}{\sqrt{1 + \sum_{j=1}^{n-1} |\partial_j h(\tilde{x})|^2}}(\tilde{x}, h(\tilde{x})) d\tilde{x}$$

$$= \int_Q v_n(\tilde{x}, h(\tilde{x})) d\tilde{x}$$



$$Q \times (0,1) \xrightarrow{\phi} \mathbb{D}$$

$$(\tilde{x}, x_n) \xrightarrow{\phi} (\tilde{x}, x_n + h(\tilde{x}) - 1)$$

$$\phi(\tilde{x}, 1) = (\tilde{x}, h(\tilde{x}))$$

$$\phi(\tilde{x}, 0) = (\tilde{x}, h(\tilde{x}) - 1)$$

v verschwindet an der Untergrenze

$$\int_{\mathbb{D}} f = \int_Q \int_0^1 f \circ \phi$$

(falls f "unten" verschwindet) " $\sqrt{g} = 1$ "

Nebenrechnung für $\sqrt{g}$, $n=2$.

$$\phi(x_1, x_2) = (x_1, x_2 + h(x_1))$$

$$D\phi(x_1, x_2) = \begin{pmatrix} 1 & 0 \\ h'(x_1) & 1 \end{pmatrix}$$

$$|\det D\phi| = 1$$

" $\sqrt{g}$ "

③

2. Fall $v = (v_1, 0, \dots, 0)$

$$\int_{\partial Q} \nabla \cdot v = \int_{\partial Q} \partial_n v_1 \stackrel{(*)}{=} \int_Q \int_0^1 (\partial_n v_1) \circ \phi$$

$$= \int_Q \int_0^1 \partial_n (v_1 \circ \phi) - (\partial_n v_1) \circ \phi \partial_n h$$

Nebenrechnung: $\partial_n (v_1 \circ \phi) = \sum_{j=1}^n (\partial_j v_1) \circ \phi \cdot \partial_n \phi_j$

$$= (\partial_1 v_1) \circ \phi \cdot 1 + (\partial_n v_1) \circ \phi \cdot \partial_n h$$

$$\stackrel{HS}{=} 0 - \int_Q \int_0^1 \partial_n (v_1 \circ \phi) \partial_n h \quad (4)$$

$$\partial_n \phi_j = 0 \quad \forall j \leq n-1$$

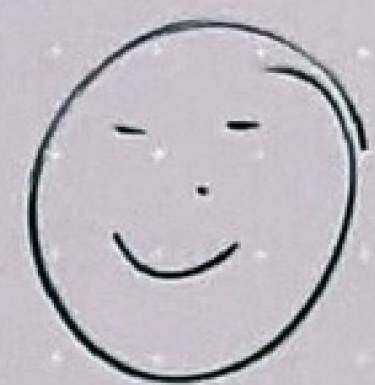
$$\partial_n \phi_n = 1$$

$$\stackrel{HS}{=} - \int_Q v_1(\tilde{x}, h(\tilde{x})) \partial_n h(\tilde{x}) d\tilde{x}$$

$$= + \int_Q v \cdot \vec{n}(\tilde{x}, h(\tilde{x})) \sqrt{\dots} d\tilde{x}$$

$$v \cdot \vec{n} = v_n \cdot (-\partial_n h) / \sqrt{\dots}$$

$$= \int_{\Gamma} v \cdot \vec{n}$$



$(0, v_2, 0, \dots, 0)$ analog

Satz von Gauß: $D \subset \mathbb{R}^n$ offen mit gutem Rand

$\vec{n} : \partial D \rightarrow \mathbb{R}^n$ der äußere Normalenvektor

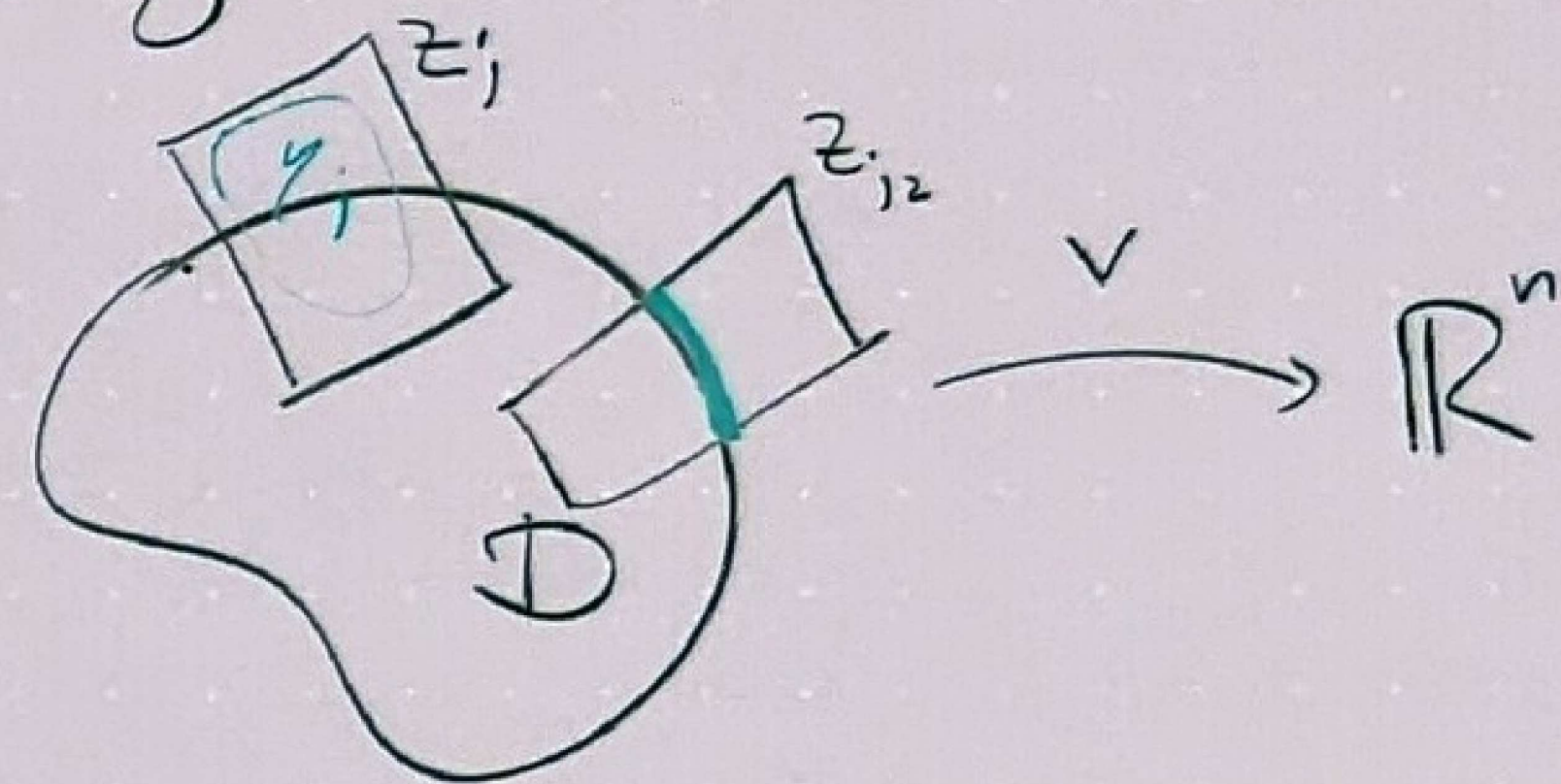
$v : \bar{D} \rightarrow \mathbb{R}^n$ ein C^1 -Vektorfeld

Dann gilt

$$\int_D \nabla \cdot v = \int_{\partial D} v \cdot \vec{n}$$

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Allgemeiner Beweis: Teilung des 1 5



So, dass

$$\sum_i \eta_i(x) = 1 \quad \forall x \in D$$

$$\eta_i : \mathbb{R}^n \rightarrow \mathbb{R}$$

η_i lebt nur in Z_i

($\eta_i = 1$ im Inneren von Z_i)

Rechnen:

$$\int_D \nabla \cdot v = \int_D \nabla \cdot \left(\sum_i \eta_i v \right) = \sum_i \int_{\substack{D \\ Z_i \cap D}} \nabla \cdot (\eta_i v) \quad \underline{\underline{\text{Unsere Rechnung}}}$$

$$= \sum_i \int_{\partial D \cap Z_i} \vec{n} \cdot \nu \, \eta_i = \int_{\partial D} \vec{n} \cdot \nu \underbrace{\sum \eta_i}_{=1}$$

6

Nochmal Flächenintegral

$$\begin{array}{ccc} Q & \xrightarrow{\varphi} & T \\ \tilde{x} & \longmapsto & (\tilde{x}, h(\tilde{x})) \end{array}$$

$$\sqrt{g} = ?$$

$$D\varphi(\tilde{x}) = \begin{pmatrix} 1 & & & 0 \\ 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & 0 & 1 \\ \partial_1 h & & & \partial_{n-1} h \end{pmatrix}$$

$$\varphi(\tilde{x}) = \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \\ h(x_1, \dots, x_{n-1}) \end{pmatrix}$$

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$$D\varphi(\tilde{x})^T = \begin{pmatrix} 1 & 0 & \partial_1 h \\ 0 & 1 & \partial_{n-1} h \end{pmatrix} \quad (7)$$

$$D\varphi^T \cdot D\varphi = \begin{pmatrix} 1 + |\partial_1 h|^2 & \partial_1 h \partial_2 h \\ \partial_1 h \partial_2 h & 1 + |\partial_{n-1} h|^2 \\ \vdots & \vdots \\ \partial_1 h \partial_{n-1} h & \partial_2 h \partial_{n-1} h \end{pmatrix}$$

$$\sqrt{g} = \sqrt{\det D\varphi^T D\varphi}$$

$$\frac{n=3}{(\tilde{x} \in \mathbb{R}^2)}$$

$$\det \begin{pmatrix} 1 + |\partial_1 h|^2 & \partial_1 h \partial_2 h \\ \partial_1 h \partial_2 h & 1 + |\partial_2 h|^2 \end{pmatrix}$$

$$= (1 + |\partial_1 h|^2)(1 + |\partial_2 h|^2) -$$

$$-|\partial_1 h|^2 |\partial_2 h|^2 = 1 + |\partial_1 h|^2 + |\partial_2 h|^2$$

$$\sqrt{g} = \sqrt{1 + |\partial_1 h|^2 + |\partial_2 h|^2}$$

Beliebiges n :

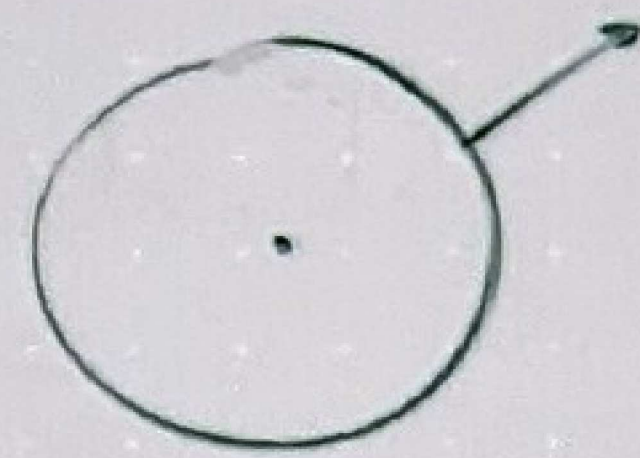
$$\sqrt{g} = \sqrt{1 + \sum_i |\partial_i h|^2}$$

Anwendungen von Gauß

(8)

$$D := B_r(0) \subset \mathbb{R}^n$$

$$\{x \in \mathbb{R}^n \mid |x| < r\}$$



$$v(x) = x$$

$$\nabla \cdot v(x) = n$$

$$\int_{B_r} \nabla \cdot v = n \operatorname{Vol}_n(B_r)$$

Gauß $\leadsto$

$$x \in \partial B_r:$$

$$\vec{n} = \frac{x}{|x|}, \quad v \cdot \vec{n} = r,$$

$$\int_{\partial B_r} v \cdot \vec{n} = r \operatorname{Vol}_{n-1}(\partial B_r)$$

Also: $\text{Vol}_n(B_r) = \frac{r}{n} \text{Vol}_{n-1}(\partial B_r)$

$n=3$:

$$\frac{4}{3} \pi r^3 = \frac{r}{3} \cdot 4\pi r^2$$

$$\pi r^2 = \frac{r}{2} \cdot 2\pi r$$

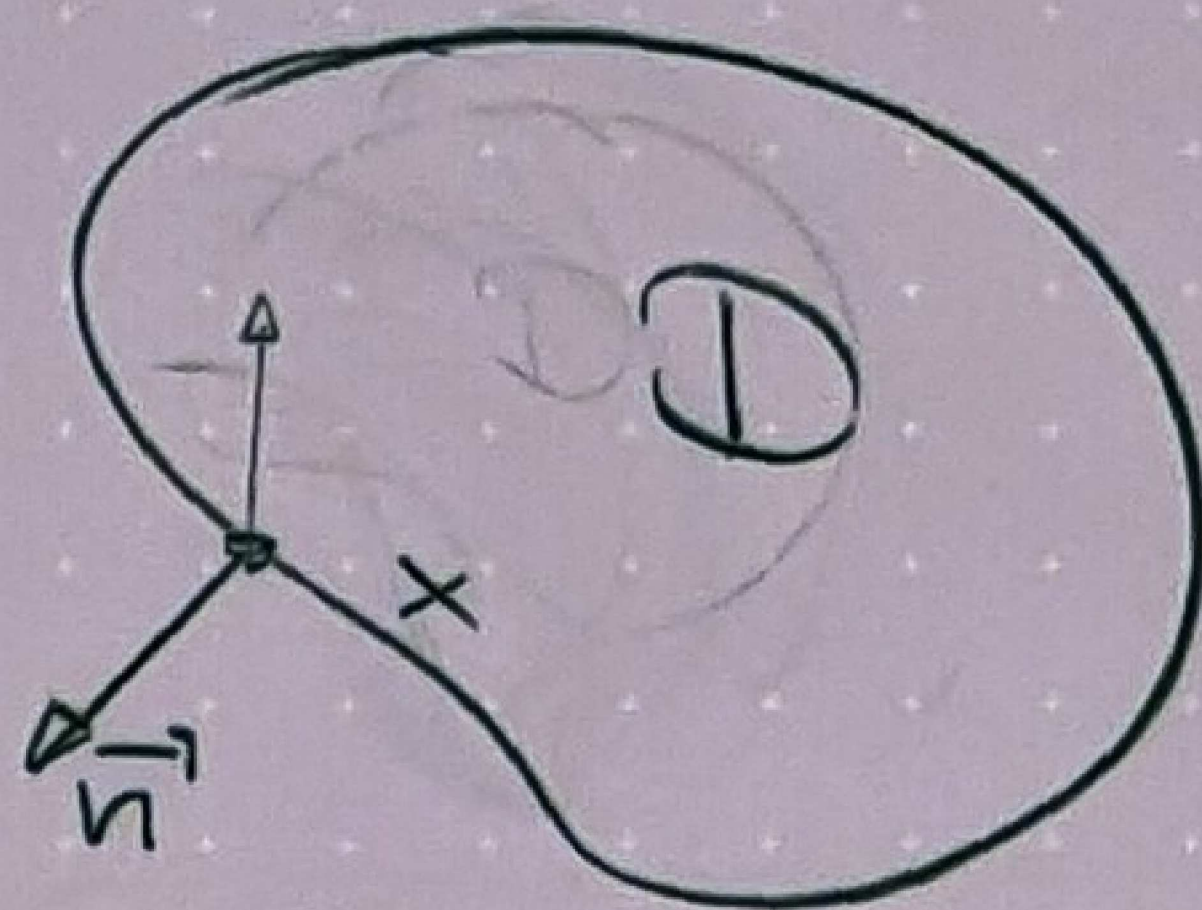
$n=2$

Auftrieb eines Körpers in

Wasser

~~~~~  $x_3 = 0$

Druck  $p(x)$



Hydrostatik  $\Rightarrow p(x) = -\rho g x_3$  (9)

Kraft im Punkt auf Körper:

$$\vec{F}(x) = -p(x) \vec{n} = \rho g x_3 \vec{n}$$

(Kraftdichte) Gesamtkraft ist

$$\int_{\partial D} \vec{F} \cdot \vec{e}_3 = \int_{\partial D} \underbrace{\rho g x_3 \vec{e}_3}_{=v} \cdot \vec{n} \stackrel{\text{Gauß}}{=} \int_D \nabla \cdot v$$

$$= \rho g \int_D 1 = \rho g \text{Vol}_n(D)$$



Gauß:

$$\int_{\partial D} \nabla v = \int_D \nabla \cdot \vec{n}$$

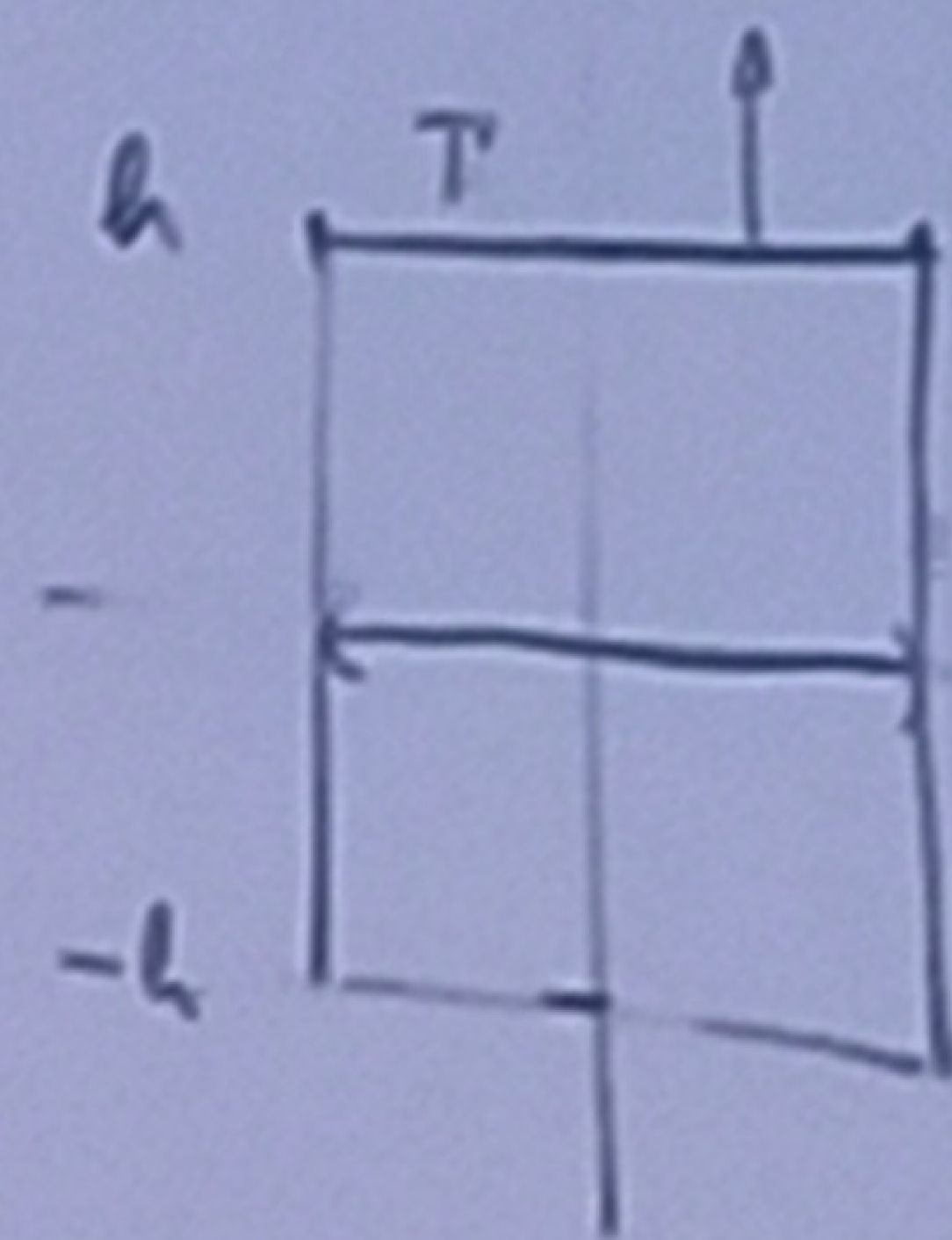


Elementares Beispiel ①

$$\mathbb{R}^2, \quad Q = (-h, h)^2$$

$$v(x) = x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\nabla v = 2 \quad \text{konst. Fld}$$



Oberseite

$$\{(x_1, x_2) \mid x_2 = h\}$$

$$\vec{n} = e_2$$



Integral über Oberseite  $T$

$$\int_T v \vec{n} = \int_{-h}^h \begin{pmatrix} x_1 \\ h \end{pmatrix} e_2 dx_1 = h \cdot 2h = 2h^2$$

Andere Seite genauso

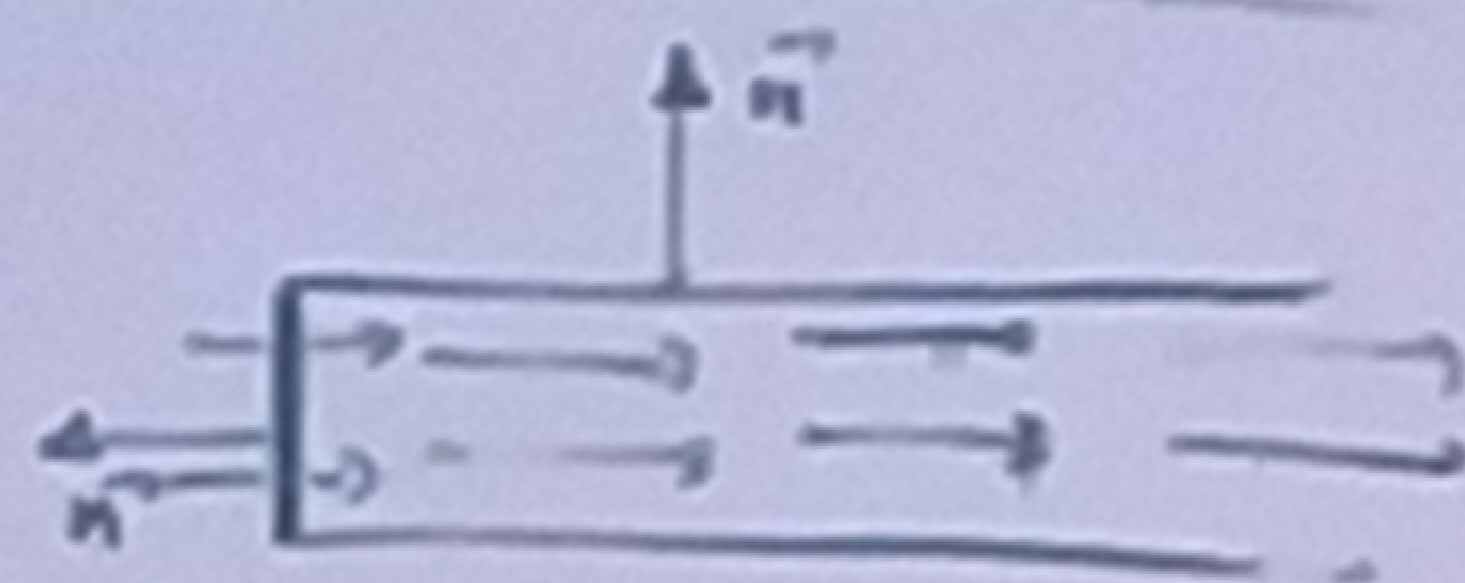
$$\int_{\partial Q} v \vec{n} = 4 \cdot 2h^2 = 8h^2$$

$$\int_Q \nabla \cdot v = \int_Q z = (2h)^2 \cdot z = 8h^2$$

✓

Warnendes Beispiel 1

(2)



$$D = (0, \infty) \times (0, 1) \subset \mathbb{R}^2$$

$$v: D \rightarrow \mathbb{R}^2$$

$$v(x) = e_1 \quad \forall x$$

$$\int_{\partial D} v \vec{n} = \int_0^1 e \cdot (-e_1) = -1$$

$$\nabla \cdot v = 0$$

Wollen immer

$D$  beschränkt!



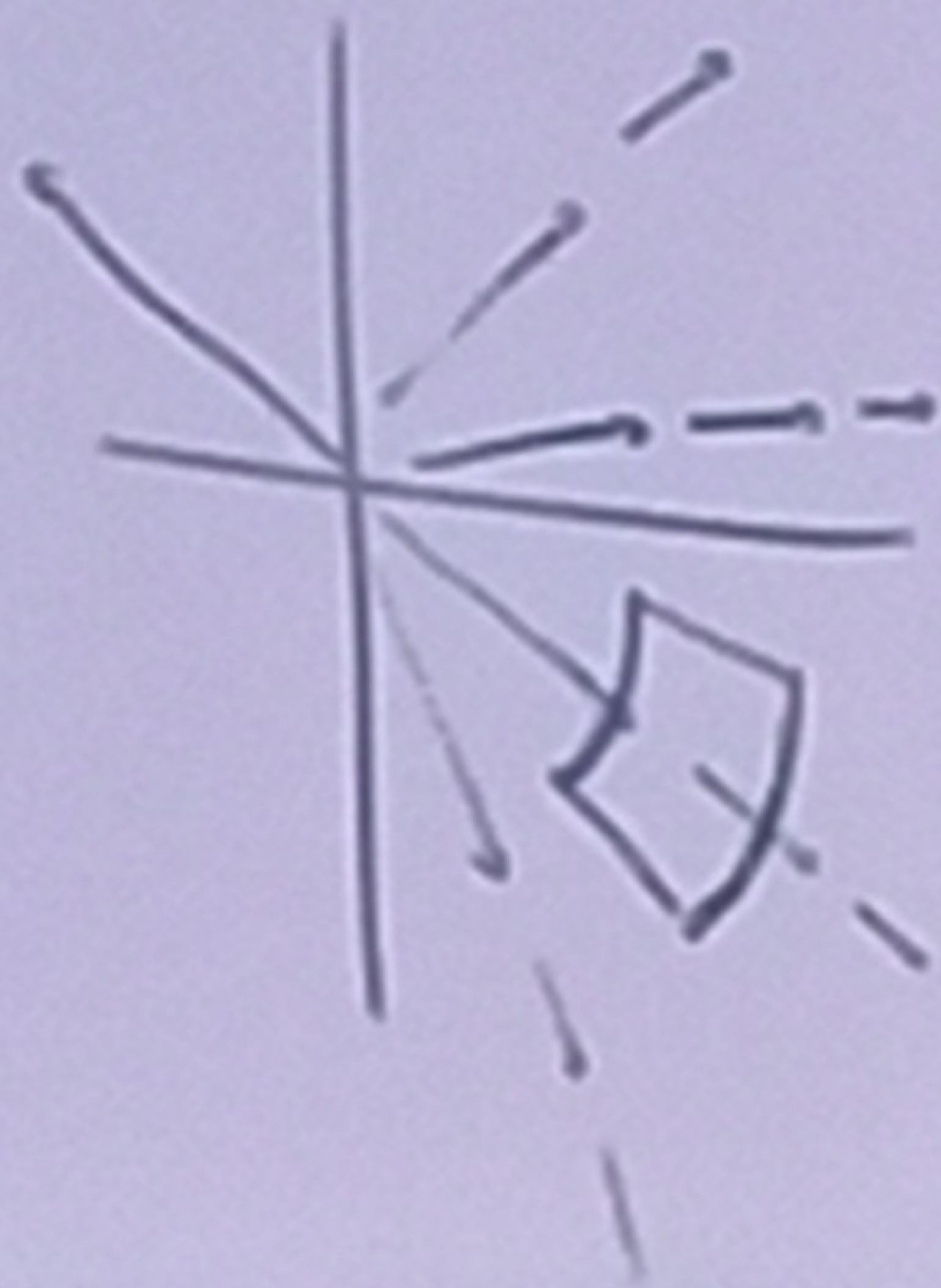
## Wasnendes Beispiel 2

$$v(x) = \frac{1}{|x|^2} \quad x \in \mathbb{R}^2$$

$$= \frac{1}{|x|^2} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\nabla v = ?$$

$$\partial_1 v_1(x) = \partial_1 \left( \frac{1}{|x|^2} x_1 \right)$$



$$= \frac{1}{|x|^2} - x_1 \cdot 2x_1 \cdot \frac{1}{|x|^4}$$

$$= \frac{1}{|x|^2} - 2x_1^2 \frac{1}{|x|^4}$$

Analog

$$\partial_2 v_1(x) = \frac{1}{|x|^2} - 2x_2^2 \frac{1}{|x|^4}$$

$$\nabla v(x) =$$

$$= 2 \frac{1}{|x|^2} - 2 \frac{|x|^2}{|x|^4}$$

$$= 0$$

③

$$\partial_1 \left( \frac{1}{|x|^2} \right) =$$

$$= \partial_1 \left( |x|^2 \right)^{-1}$$

$$= - \frac{1}{|x|^3} 2x_1$$

$$|x|^2 = x_1^2 + x_2^2$$