35 Potentialgl

①

$$\Delta = \sum_{j=1}^{n} \partial_j^2$$

$\Delta u = f$ (Potentialgl)

$\Delta u = 0$ (u harmonisch)

Harmonische Fkt: $u(x) = c + b \cdot x$ $\quad c \in \mathbb{R}$, $b \in \mathbb{R}^n$

$x_1 x_2$, $\quad x_1^2 - x_2^2$,

$n = 3$ $\quad u(x) = c_1 x_1 + x_2^2 - x_3^2$

in $\mathbb{R}^2$ $(\cong \mathbb{C})$:

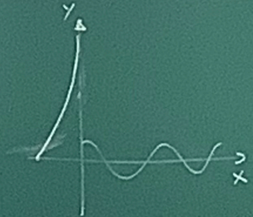
h holomorph

$u := \operatorname{Re} h$ harm.

$u := \operatorname{Im} h$ — " —

$z \mapsto z^2$, $u := \operatorname{Re} z^2 = x^2 - y^2$

$z \mapsto e^z$, $u := \operatorname{Re} e^z = \operatorname{Re}(e^x e^{iy}) = e^x \cos y$



S

Bisher: 1D: Lsg. mit Integral

Darstellung "D"

Existenz "E"

Quader: $(0,\pi)^2$

D & E

$$u(x) = \sum_{k,l\in\mathbb{N}_0} a_{k,l}\,(\sin kx_1)(\sin lx_2)$$

f mit $b_{k,l}$ darstellen ...

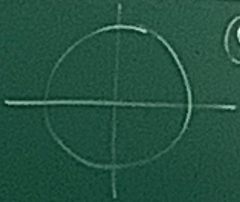
$$a_{k,l} = \frac{-b_{k,l}}{k^2+l^2}$$

Gilt auch in $\mathbb{R}^n$:

$$\prod_{j=1}^{n} (\sin k_j x_j)$$

$$\lambda = \sum_j k_j^2$$

Kugeln



$n = 2$. $\Omega := B_1(0) \subset \mathbb{R}^2$

Laplace-Op. in Polarkoord.:

$$\Delta u = \frac{1}{r}\partial_r(r\partial_r u) + \frac{1}{r^2}\partial_\varphi^2 u$$

$u(x_1, x_2)$ $u(r, \varphi)$

(mit part. Integr.: $\int \Delta u \, \psi = -\int \nabla u \nabla \psi = -\int (\partial_r u \, e_r + \frac{1}{r}\partial_\varphi u \, e_\varphi, \partial_r \psi \, e_r + \frac{1}{r}\partial_\varphi \psi \, e_\varphi)$ ⑤

$\underbrace{(\partial_r u \, e_r + \frac{1}{r}\partial_\varphi u \, e_\varphi, \partial_r \psi \, e_r + \frac{1}{r}\partial_\varphi \psi \, e_\varphi)}_{=\partial_r u \, \partial_r \psi + \frac{1}{r^2}\partial_\varphi u \, \partial_\varphi \psi =: [\quad]}$

$$= -\int_0^1 \int_0^{2\pi} [\quad] \, r \, d\varphi \, dr$$

$$= \int\int \frac{1}{r}\partial_r(r\partial_r u) \, \psi \, r \, dr \, d\varphi + \ldots)$$

Damit finden wir nun Δu in $B_1(0)$

Ansatz: $u(r,\varphi) = g(r)(\sin\varphi)$

$$\rightsquigarrow \frac{1}{r^2}\partial_\varphi^2 u = -\frac{1}{r^2}(\sin\varphi) g(r)$$

Ziel: Löse $0 = \Delta u = \left[\frac{1}{r}\partial_r(r\partial_r g(r)) - \frac{1}{r^2}g(r)\right]\sin\varphi$

Rand: Kanonisch ist $u(1,\varphi) = \sin\varphi = x_2$

$\rightsquigarrow \quad u(x) = x_2$

In Polarkoord.: $u(r,\varphi) = r\sin\varphi$

Also: $g(r) = r$

<u>Test:</u>

$$\frac{1}{r}\underbrace{\partial_r\Big(\underbrace{r\,\underbrace{\partial_r \overbrace{r}^{g(r)=r}}_{=1}}_{r}\Big)}_{=1} = \frac{1}{r}$$

$$\frac{1}{r^2}g(r) - \frac{1}{r^2}r = \frac{1}{r} \quad \checkmark$$

Finde allgemeine Lösungen:

$k \in \mathbb{N}$ Ansatz: $u(r,\varphi) = g_k(r)\,(\sin k\varphi)$

$$0 = \Delta u \iff \frac{1}{r}\partial_r\left(r\,\partial_r g_k(r)\right) = \frac{1}{r^2}k^2\,g_k(r)$$

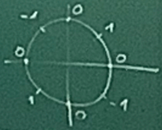
$g_k(r) = r^k$:

$$\frac{1}{r}\partial_r\left(r\,\partial_r\left(r^k\right)\right) = \frac{1}{r}\partial_r\left(k r^k\right) = k^2 r^{k-2}$$

Bsp: $k = 2$: $\quad u(r,\varphi) = r^2 \sin(2\varphi) = \operatorname{Im} z^2 = 2 x_1 x_2$

Betrachte:

$\left(\begin{array}{l} z = r e^{i\varphi} \\ \Rightarrow z^2 = r^2 e^{2i\varphi} \end{array} \right)$



$$\Delta u = 0 \quad \text{in } B_1(0) \subset \mathbb{R}^2$$

$$u\big|_{\partial B_1} = g, \qquad g : \varphi \mapsto g(\varphi).$$

Schreibe $g(\varphi) = \sum_{k \in \mathbb{Z}} g_k e^{ik\varphi}$

Dirichlet-Pb. auf $B_1(0)$

(Fourier-Reihe von g)

Lösung ist

$$u(r,\varphi) = \sum_{k \in \mathbb{Z}} c_k e^{ik\varphi} r^{|k|}$$

("L" & "D")

__Reell:__

$$g(\varphi) = \sum_{k \in \mathbb{N}} a_k \cos k\varphi + b_k \sin k\varphi \implies u(r,\varphi) = \sum_{k \in \mathbb{N}} a_k \cos k\varphi \, r^k + b_k \sin k\varphi \, r^k$$

Kugeln, bel. Dim. $n \geq 2$ ⑩

$r = |x|$

$\xi \in S^{n-1} = \partial B_1^n(0)$

Info 1 Laplace-Op. ist, mit $x = r\xi$

$$\Delta u = \frac{1}{r^{n-1}} \partial_r \left(r^{n-1} \partial_r u \right) + \frac{1}{r^2} \Delta_{LB} u$$

Δ_{LB} = Laplace-Beltrami-Op., enth. nur ξ-Abl.

Indes.: Für $u(r,\varphi) = g(r)$ (11)

gilt $\Delta u(r,\varphi) = \frac{1}{r^{n-1}} \partial_r \left(r^{n-1} \partial_r g(r) \right)$

Info 2: $-\Delta_{LB}$ hat Eigenfunktionen (3D: Kugelflächenfkt, spherical harmonics $Y_k : S^{n-1} \to \mathbb{R}$)

mit Eigenwerten λ_k:

$$-\Delta_{LB} Y_k = \lambda_k Y_k$$

Finde harm. Fkt.

(12)

Info 3. $u(r,\xi) := r^k Y_k(\xi)$ ist harmonisch

$$\frac{1}{r^{n-1}}\partial_r\left(r^{n-1}\partial_r r^k\right) = \frac{1}{r^{n-1}}\partial_r\left(k\, r^{n+k-2}\right)$$

$$= k\,(n+k-2)\, r^{n+k-3-(n-1)}$$

$$= k\,(n+k-2)\, r^{k-2}$$

Also:

$$\Delta u = k(n+k-2)\, r^{k-2}\, Y_k(\xi) + r^{k-2}\, \Delta_{LB} Y_k(\xi)$$

$$\Delta u = \left[k(n+k-2) - \lambda_k\right] r^{k-2} Y_k(\xi)$$

$$\overset{!}{=} 0 \quad \Longleftrightarrow \quad \lambda_k = k(n+k-2)$$

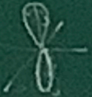
Das ist der k-te EW von $-\Delta_{LB}$

$Y_0 \equiv \text{const}$

$\lambda_0 = 0$

$\rightsquigarrow$ s-Orbital

Y_1 : 

bzw  p-Orbital

Y_2 : d-Orbital

$\vdots$

Damit auch in bel. Dim: $\exists$ Lsg. zu $\begin{aligned} \Delta u &= 0 \text{ in } B_1(0) \\ u\big|_{\partial B_1(0)} &= g \end{aligned}$ (14)

Damit 1D ✓ Quader ✓ Kugeln ✓

Jetzt: Ganzraum: $\boxed{\mathbb{R}^n}$

Zu $f: \mathbb{R}^n \to \mathbb{R}$

suche $u: \mathbb{R}^n \to \mathbb{R}$

$$\Delta u = f$$

Newton-Potential

$n \geq 3$:

$$\Phi_n(x) := c_n \cdot \frac{1}{|x|^{n-2}}$$

$n = 2$:

$$\Phi_2(x) := c_2 \log|x|$$

$$\nabla \phi_n(x) = \ ?$$

$$\nabla \left(\frac{1}{|x|^{n-2}} \right) = \nabla \left(\left(|x|^2 \right)^{-\frac{n-2}{2}} \right) = -\frac{n-2}{2} \left(|x|^2 \right)^{-\frac{n-2}{2}-1} 2x$$

$$|x|^2 = x \cdot x$$

$$\nabla |x|^2 = 2x$$

$$= -(n-2)\, |x|^{-n}\, x$$

$$= -(n-2) \frac{1}{|x|^{n-1}} \frac{x}{|x|}$$