40 P.

HöMa IV

①

Moodle ✓

Tafelfotos ⟶

Übungen

Inhalt der Vorlesung

39 Gruppentheorie (B) (2)

40 Stabilität von Lösungen (C)

35 Potentialgl. $-\Delta u = f$

36 Wärmeleitungsgl. $\partial_t u - \Delta u = f$

37 Wellengl. $\partial_t^2 u - \Delta u = f$

3 partielle Diffgl. } PDG-Block (A)

38 Distributionen

PDG - Block

3 Gleichungen

$\boxed{E}$ $-\Delta u = f$

$\boxed{P}$ $\partial_t u - \Delta u = f$

$\boxed{H}$ $\partial_t^2 u - \Delta u = f$

3 Fragestellungen

(L) Lösungstheorie

(D) Darstellungsformeln

(E) Eigenschaften

3 wichtige "Gebietsarten"

- 1-dimensional
- $\mathbb{R}^n$ (Ganzraum, $n \geq 2$)
- $\Omega \subset \mathbb{R}^n$

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$$\Delta = \sum_{j=1}^{n} \partial_j^2 = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$$

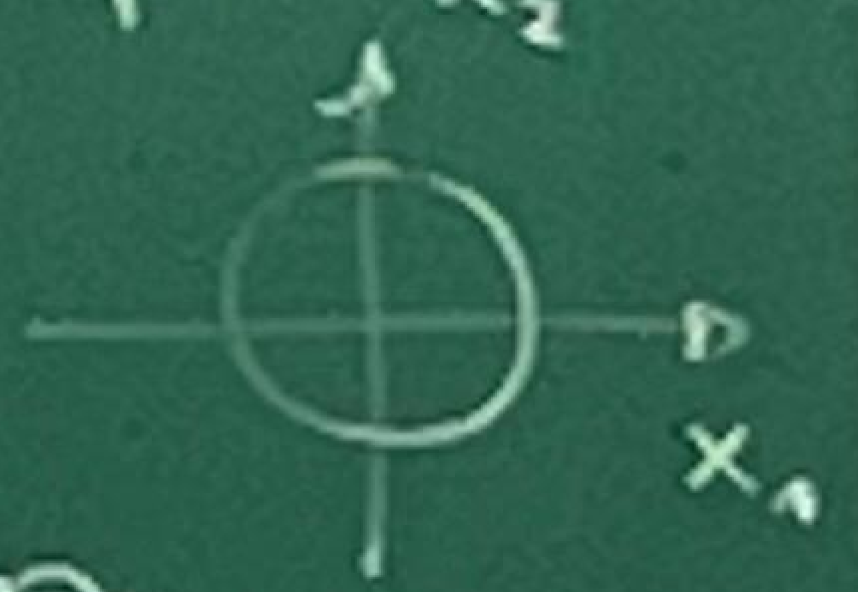
Elliptisch: $\Delta u = f$ $\longrightarrow$ $n = 2$

$\frac{\partial}{\partial x_j} \rightsquigarrow x_j$

$f \rightsquigarrow 1$

u weglassen

$x_1^2 + x_2^2 = 1$

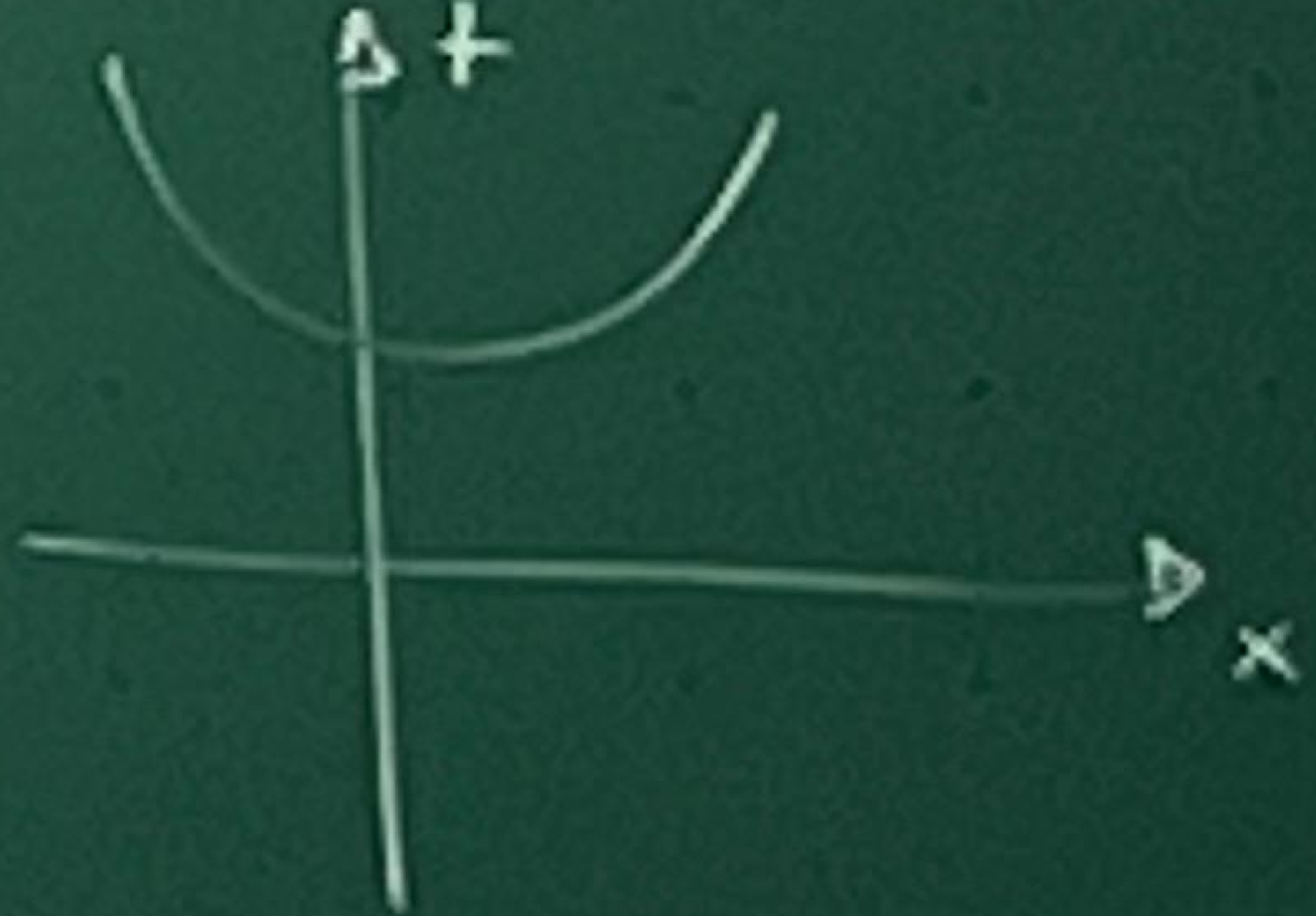


$\hookrightarrow$ Ellipsen

$a_1 \partial_1^2 u + a_2 \partial_2^2 u = f$ $\longrightarrow$ $a_1 x_1^2 + a_2 x_2^2 = 1$

Parabolisch

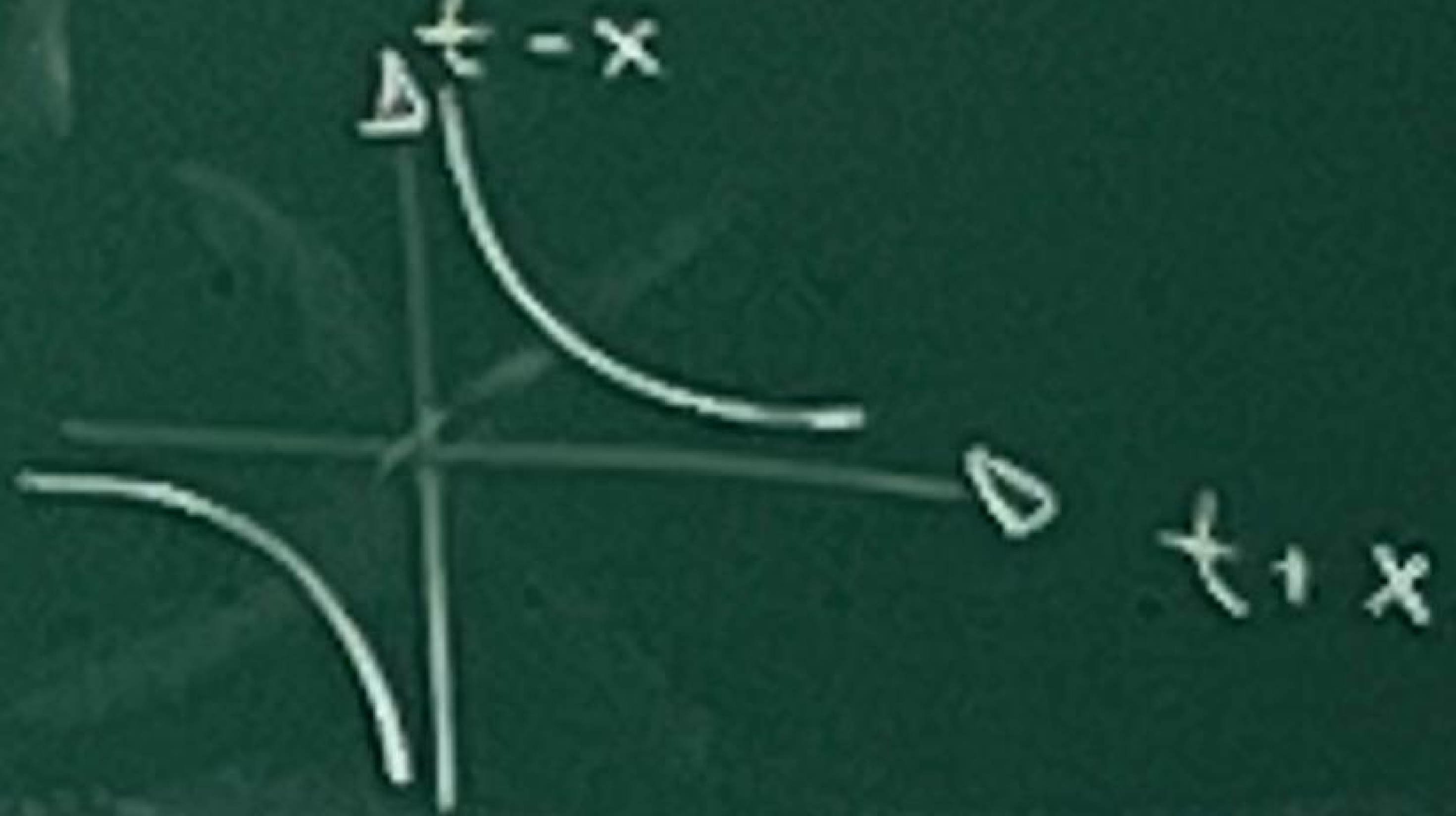
$$\partial_t u - \Delta u = f \xrightarrow{n=1} t - x^2 = 1$$



Hyperbolisch:

$$\partial_t^2 u - \Delta u = f \longrightarrow t^2 - x^2 = 1$$

$$(t+x)(t-x)$$



35 Die Potentialgl.

$$\boxed{\Delta u = f}$$

<u>Gegeben</u>: $f : \Omega \to \mathbb{R}$ (oder $\mathbb{C}$)

Gebiet $\Omega \subset \mathbb{R}^n$

<u>Gesucht</u> $u : \Omega \to \mathbb{R}$ (oder $\mathbb{C}$)

n=1, eindimensionale Gl.

$f=0$: $\partial_x^2 u = 0$

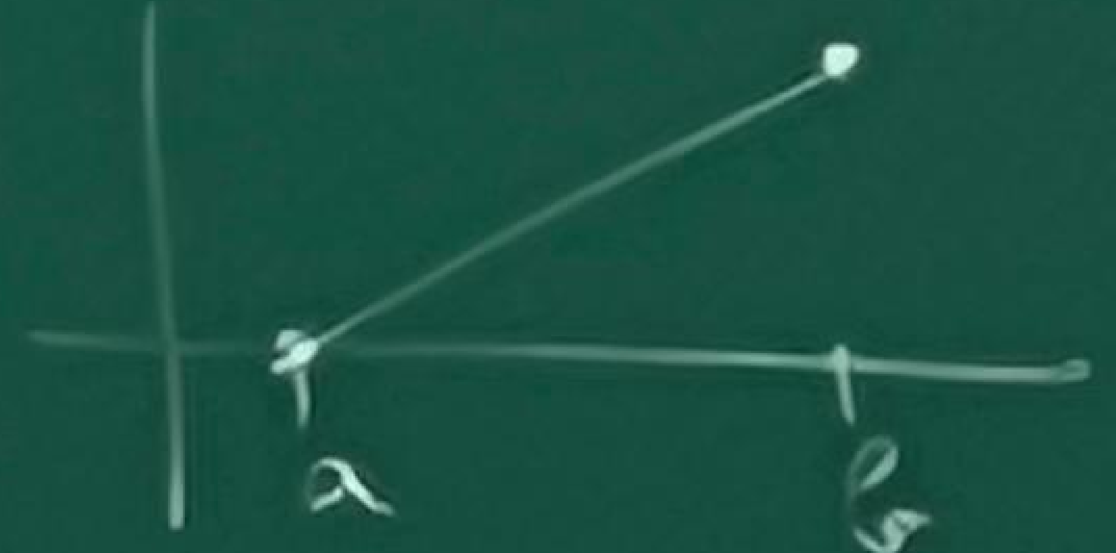
$u: I \to \mathbb{R}$, $u: x \mapsto u(x)$

$I \subset \mathbb{R}$

Lösung $u(x) = c + d x$

Randbed.: Dirichlet: $u(a) = 0$

$u(b) = 1$



Lsg: $c + d \cdot a = 0$

$c + d\, b = 1$

$bc - ac = -a \rightsquigarrow c = \frac{-a}{b-a}, \; d = \frac{1}{b-a}$

Ergebnis: $\Delta u = 0$ & Dirichlet-Bed $\rightsquigarrow$ $\exists!$ Lsg.

(auch: $\Delta u = f$)

Neumann-Randbed: $\partial_x u(a) = 0$, $\partial_x u(b) = 0$ $\rightsquigarrow$ Lsg: $u(x) = c$ $\forall x$

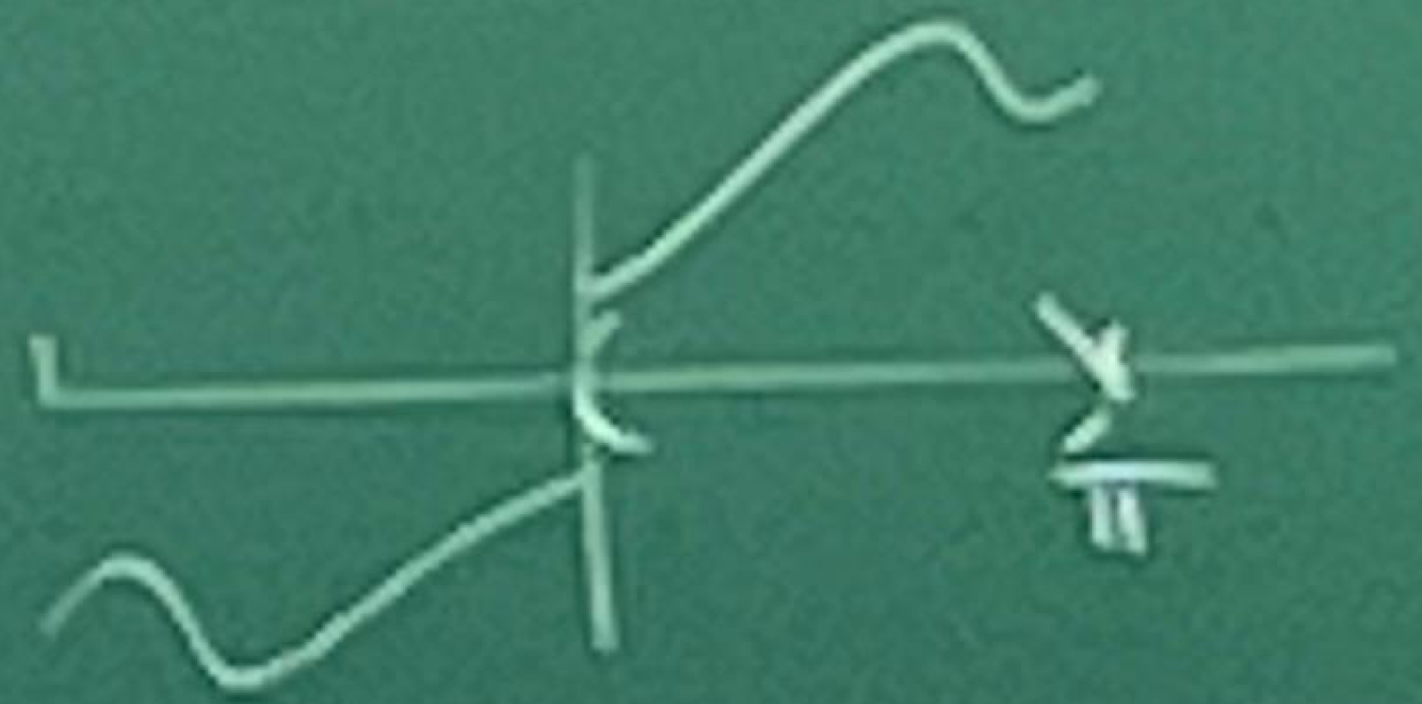
(keine Eind.)

$\partial_x u(a) = 0$, $\partial_x u(b) = 1$ $\rightsquigarrow$ keine Lsg.

Andere (schlechte?) Idee: 10

Löse $\Delta u = f$ auf $(0,\pi)$

mit $u(0) = u(\pi) = 0$



Schreibe f als Fourier-Reihe: $f(x) = \sum_{k=1}^{\infty} b_k \sin(kx)$

Lösung ist $u(x) = \sum_{k=1}^{\infty} a_k \sin(kx)$ mit $a_k = \frac{b_k}{-k^2}$ "D"

NR: $\Delta[\sin kx] = \partial_x^2[\sin kx] = -k^2[\sin kx]$

Wie geht es einfacher?

$\Delta u = f$, also $\partial_x^2 u = f$, also $\partial_x(\partial_x u) = f \implies \partial_x u(x) = c_1 + \int_{x_0}^{x} f(s)\,ds$

$$\Rightarrow u(x) = c_0 + \int_{x_0}^{x}\left[c_1 + \int_{x_1}^{t} f(s)\,ds\right]dt$$

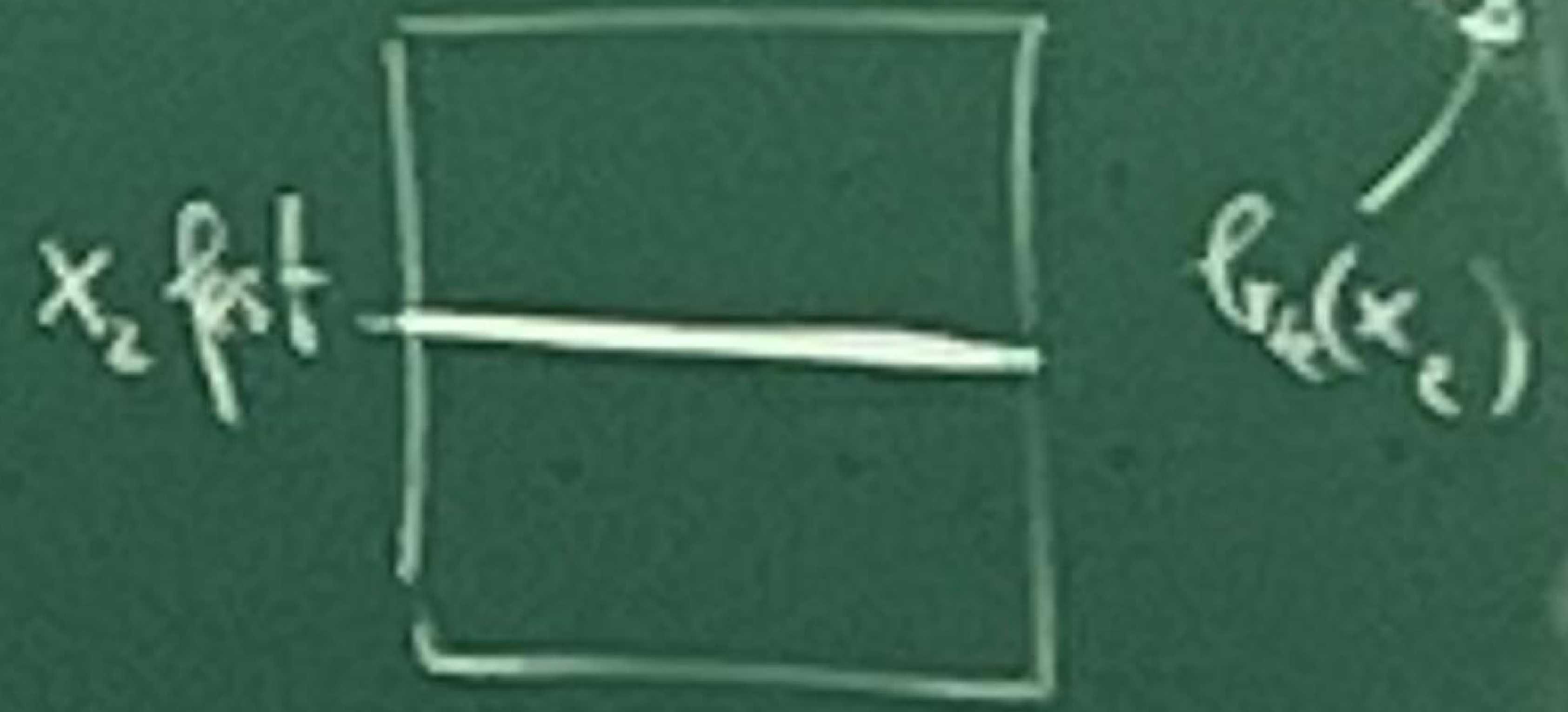
"D"

$n \in \mathbb{N}$ beliebig, Quader

<u>Bsp</u>: $n = 2$, $\Omega = (0,\pi)^2$, $\Delta u = f$, $u|_{\partial\Omega} = 0$

$$f(x_1,x_2) = \sum_{\substack{k\in\mathbb{N}\\ l\in\mathbb{N}}} b_{k,l} \sin(kx_1)\sin(lx_2)$$

$f(\cdot,x_2) = \sum_k \dots$



1. Schritt: $f(x_1,x_2) = \sum_{k\in\mathbb{N}} b_k(x_2)\sin(kx_1)$

(13)

Dann: $b_k(x_2) = \sum_{\ell} b_{k,\ell} \sin(\ell x_2)$, Einsetzen $\square$

$$\Delta\left[(\sin kx_1)(\sin \ell x_2)\right] = (-k^2 - \ell^2)\left[(\sin kx_1)(\sin \ell x_2)\right]$$

Lsg ist $u(x_1, x_2) = \sum_{\substack{k\in\mathbb{N}\\ \ell\in\mathbb{N}}} a_{k,\ell} (\sin kx_1)(\sin \ell x_2)$ mit $a_{k,\ell} = \dfrac{b_{k,\ell}}{-(k^2+\ell^2)}$

"D" "L" $\exists!$ Lsg

$\Omega \subset \mathbb{R}^n$ beschr.

Info: $\exists$ Eigenfkt. von Δ,

$\psi_1, \psi_2, \psi_3, \dots$

$-\Delta \psi_k = \lambda_k \psi_k$

$\lambda_k > 0 \ \forall k$

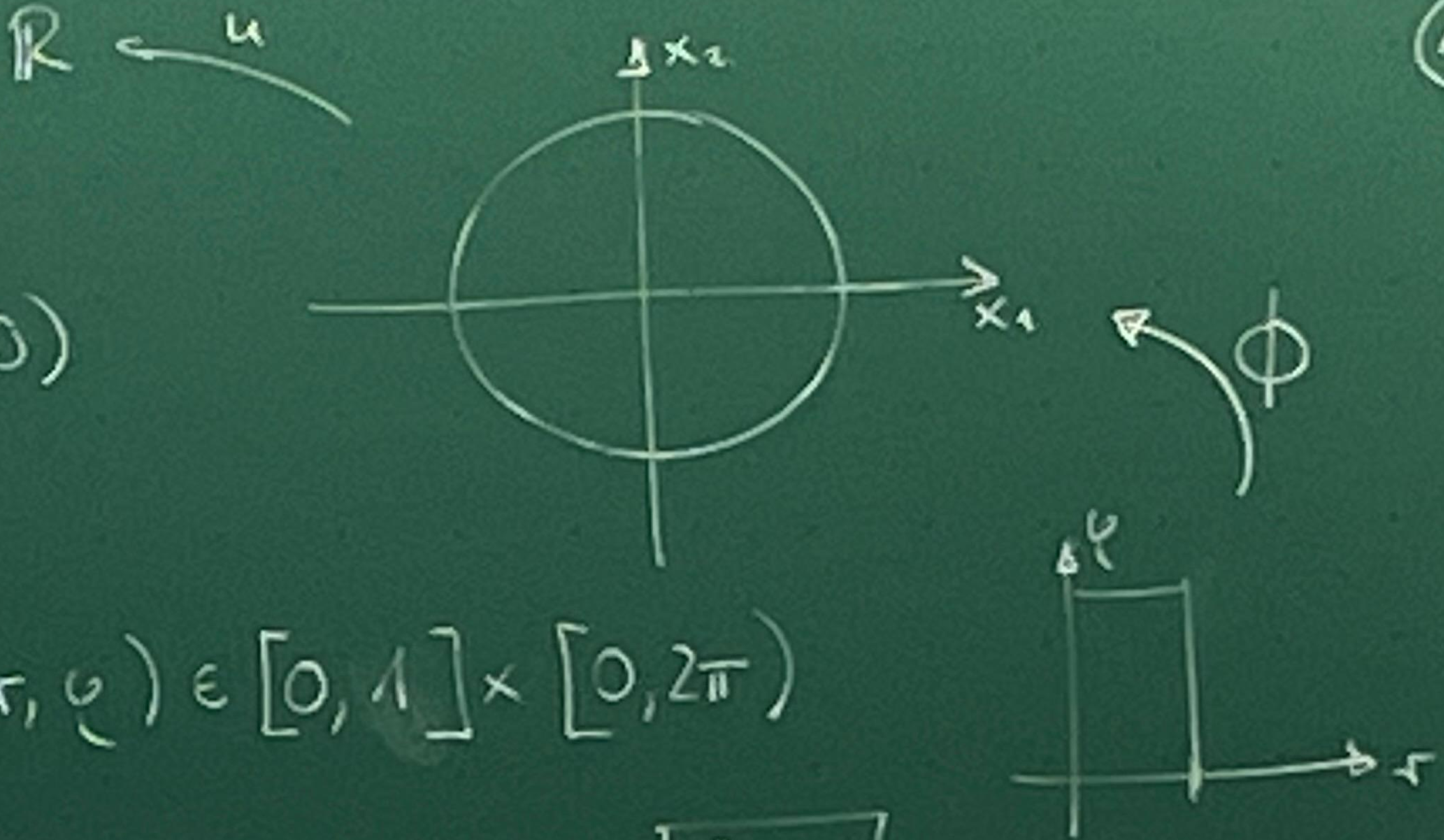
$\forall f \in L^2(\Omega)$

$\exists (b_k)_{k \in \mathbb{N}}$: $f = \sum_k b_k \psi_k$

$u = \sum_k a_k \psi_k$ mit $a_k = \frac{b_k}{-\lambda_k}$

Bilden Basis

Kugeln



$n=2, \quad \Omega = B_1(0)$

$u = u(x_1, x_2)$

Polarkoordinaten: $(r, \varphi) \in [0,1] \times [0, 2\pi)$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \end{pmatrix}$$

Umgekehrt: $r = \sqrt{x_1^2 + x_2^2}$

$\varphi = \arg(x_1 + i x_2)$

u in Polarkoord.

"$\cancel{u(r,\varphi)}$"

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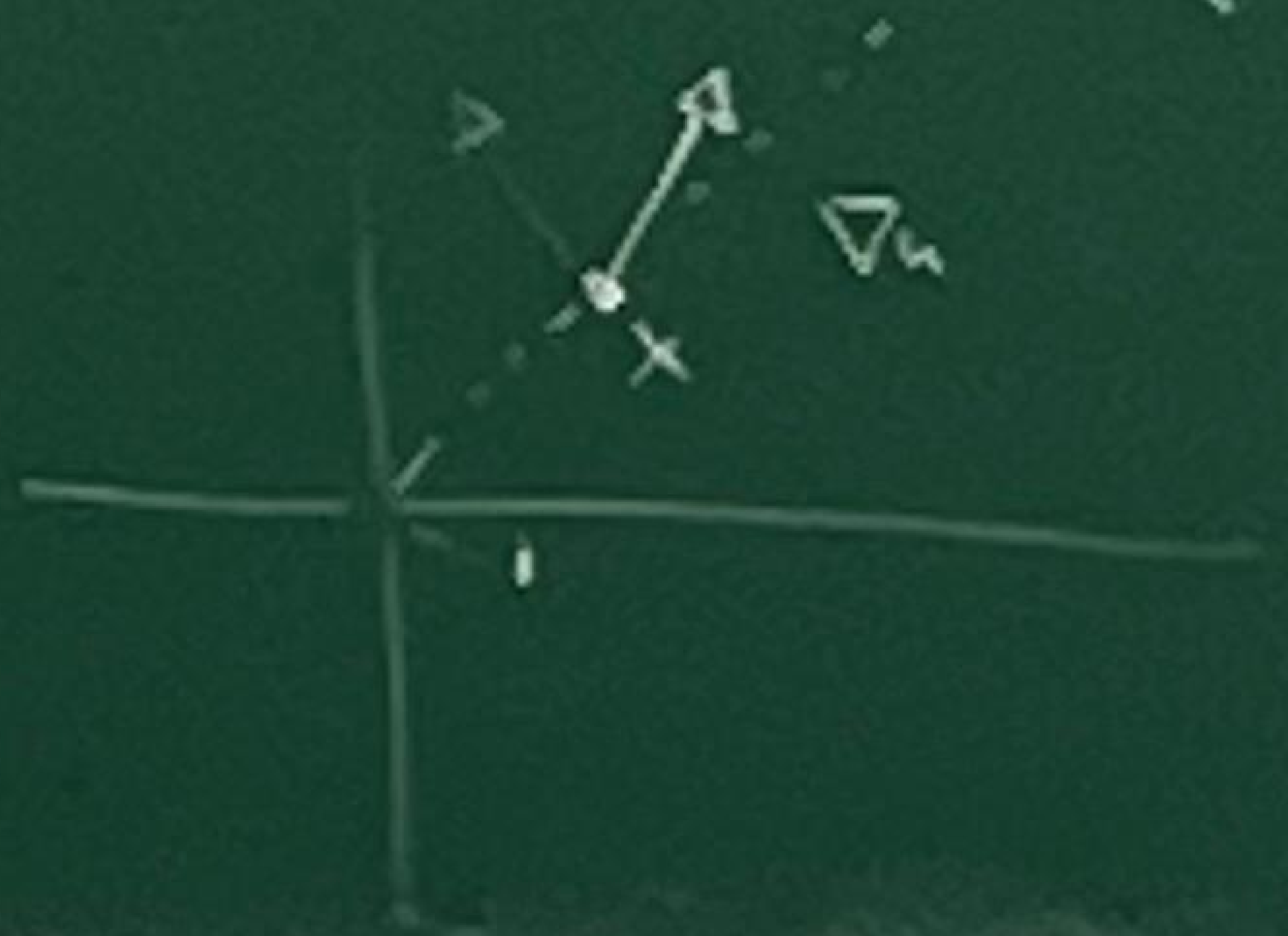
$$\nabla U = \begin{pmatrix} \partial_r U \\ \partial_\varphi U \end{pmatrix}$$ nicht nützlich!

$$U(r,\varphi) = u(x_1,x_2), \qquad U = u \circ \Phi$$

$$\nabla u(x_1,x_2) = \begin{pmatrix} \partial_1 u \\ \partial_2 u \end{pmatrix}(x_1,x_2)$$



$$= \partial_r U(r,\varphi)\,\vec{e}_r + \frac{1}{r}\partial_\varphi U(r,\varphi)\,\vec{e}_\varphi$$

Laplace-Operator in Polarkoord.:

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$$(\Delta u)(\phi(r,\varphi)) = \Big(\underset{\uparrow \atop \frac{1}{r^{n-1}}}{\frac{1}{r}}\partial_r\big(\underset{\uparrow \atop r^{n-1}}{r}\partial_r U\big) + \frac{1}{r^2}\partial_\varphi^2 U\Big)(r,\varphi)$$

Woher?

$\psi = \varphi \circ \phi$

$$\Delta(u\circ\phi) = \Delta U = \partial_r^2 U + \partial_\varphi^2 U$$

nicht möglich!

$$-\int_{\mathbb{R}^2}\Delta u\,\varphi = \int_{\mathbb{R}^2}\nabla u\,\nabla\varphi = \int_0^\infty\int_0^{2\pi}\Big(\underbrace{\partial_r U}_{\frac{1}{r}\partial_r(r\partial_r U)}\vec e_r + \frac{1}{r}\partial_\varphi U\vec e_\varphi\Big)\cdot\Big(\partial_r\psi\,\vec e_r + \frac{1}{r}\partial_\varphi\psi\,\vec e_\varphi\Big)\,\underset{r^{n-1}}{r}\,d\varphi\,dr$$